Bend number of cocomparability graphs

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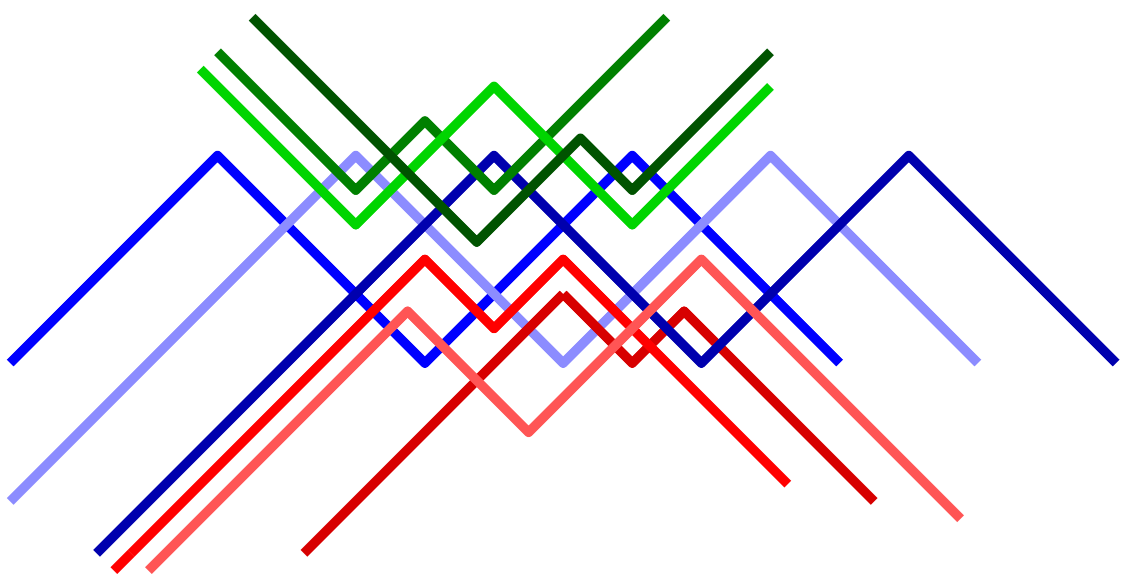
Felix Schröder, Peter Stumpf, Pavel Valtr

GD 2025



Cocomparability graphs

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Poset $(P, \preceq)$

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Cocomparability graph on P:

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- $V(G) = P$

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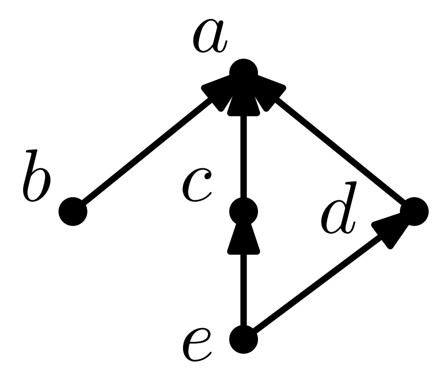

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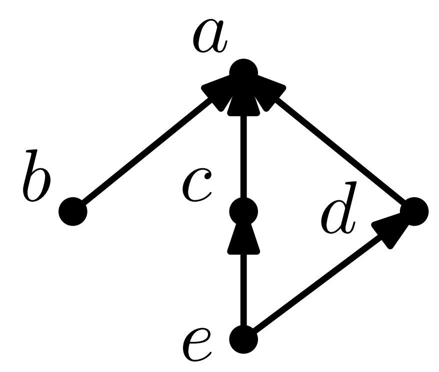

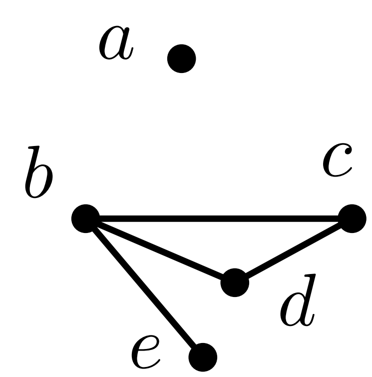

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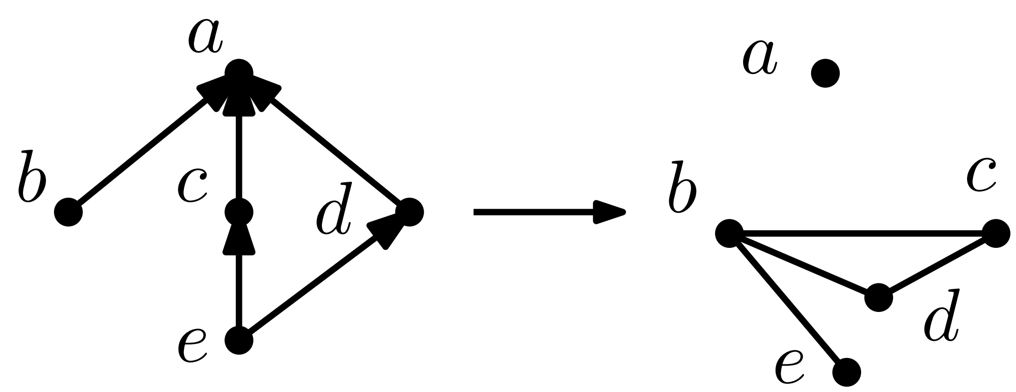

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Thm: [Golumbic, Rotem and Urrutia] G is a cocomporability graph of some poset $(P, \preceq)$ iff G is an intersection graph of x-monotone curves with endpoints on vertical lines.

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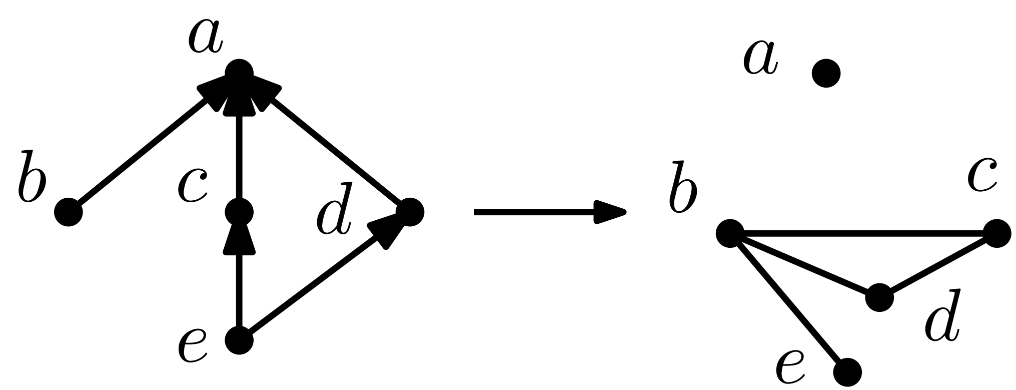

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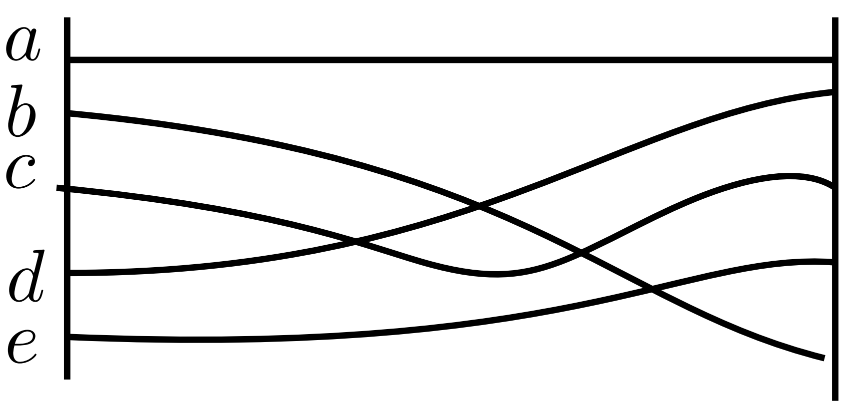

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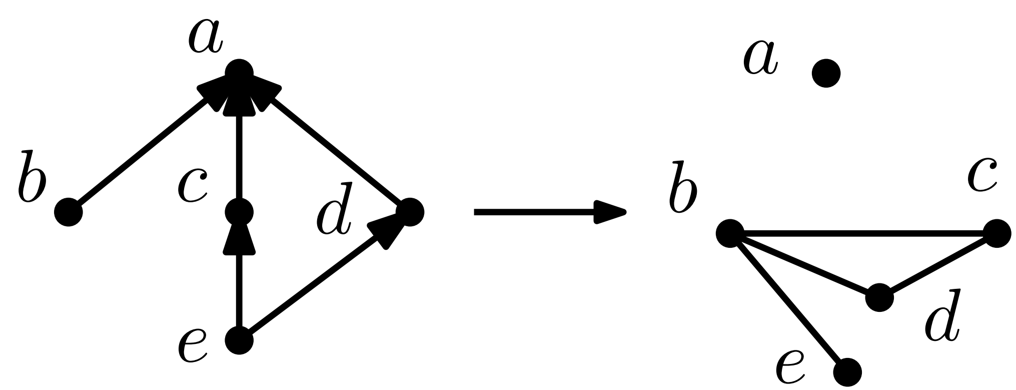

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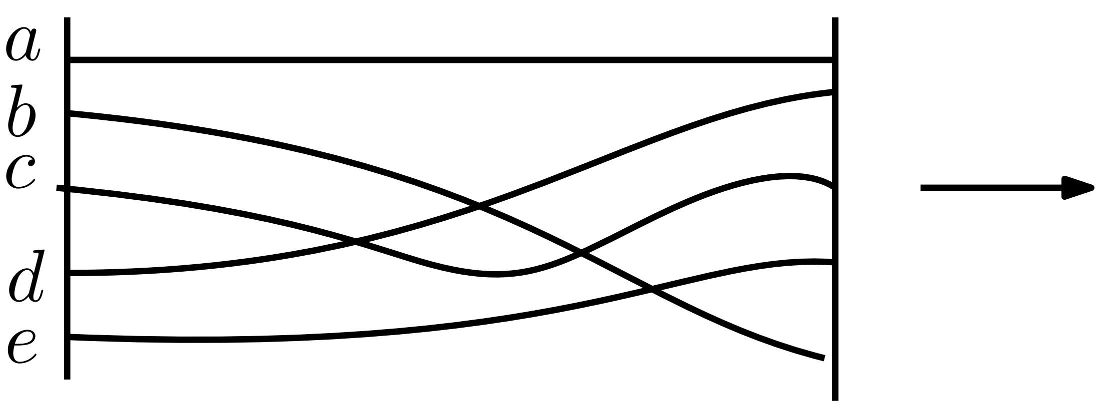

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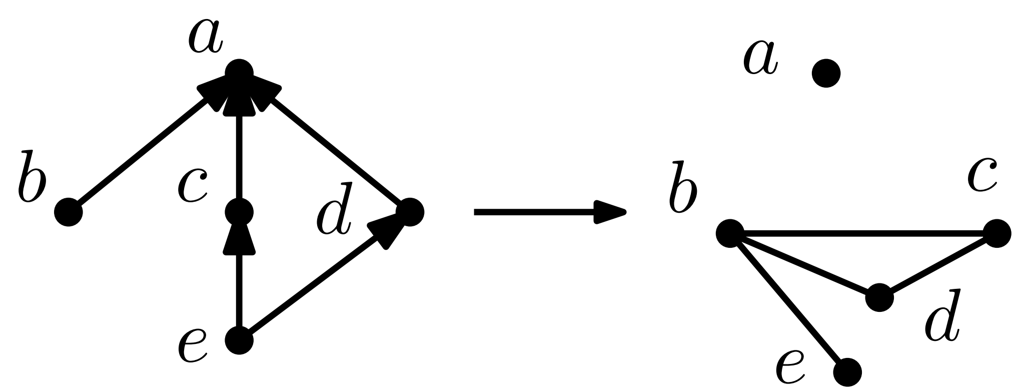

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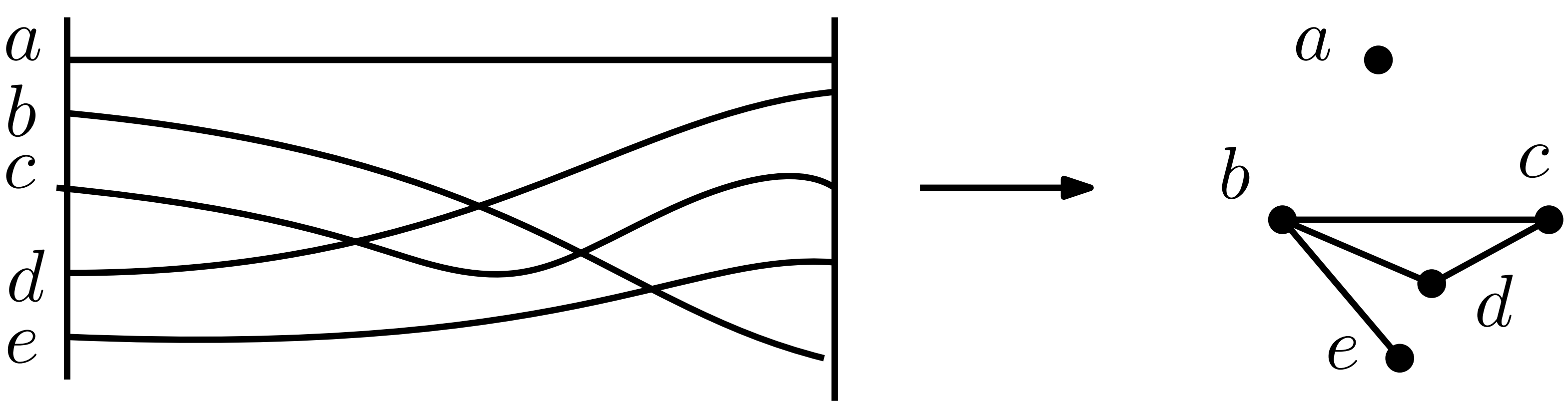

Bend-bounded representations

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- **Diagonal**: only two slopes (w.l.o.g ± 1)

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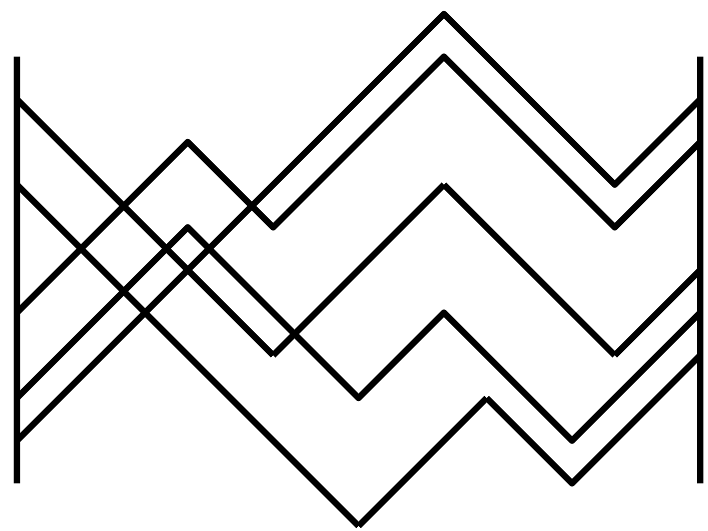

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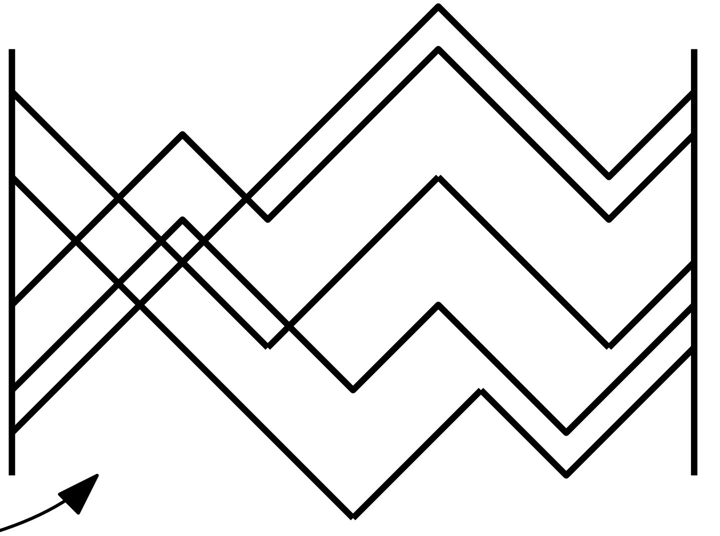

Recognition in NP!

Diagonal setting

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Price of *nice* representations $=$ more things to distinguish

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How many bounding lines?

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exactly/at most k bends?

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All curves start with the same slope?

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yes = **synchronous**

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All curves start with the same slope?

yes = **synchronous**

no = **asynchronous**

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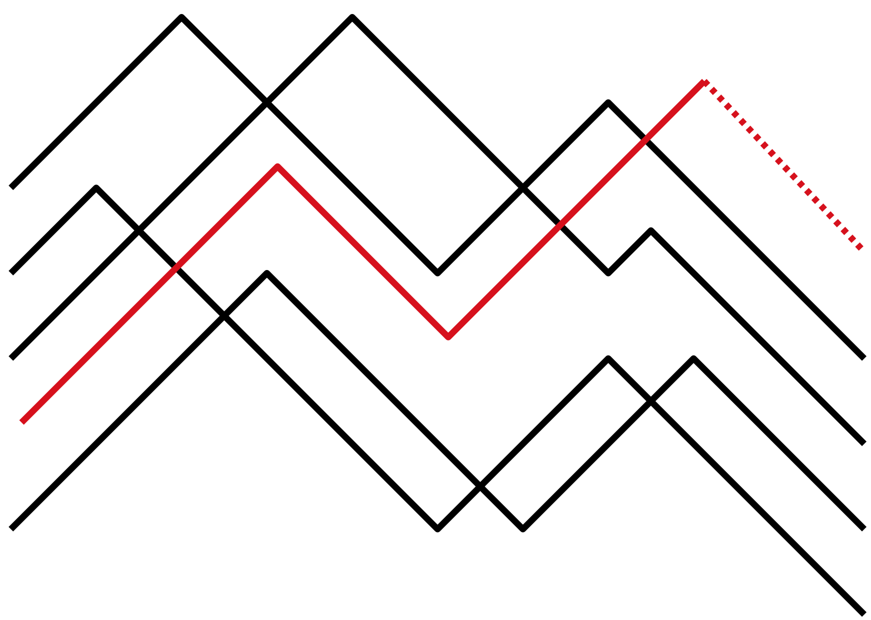

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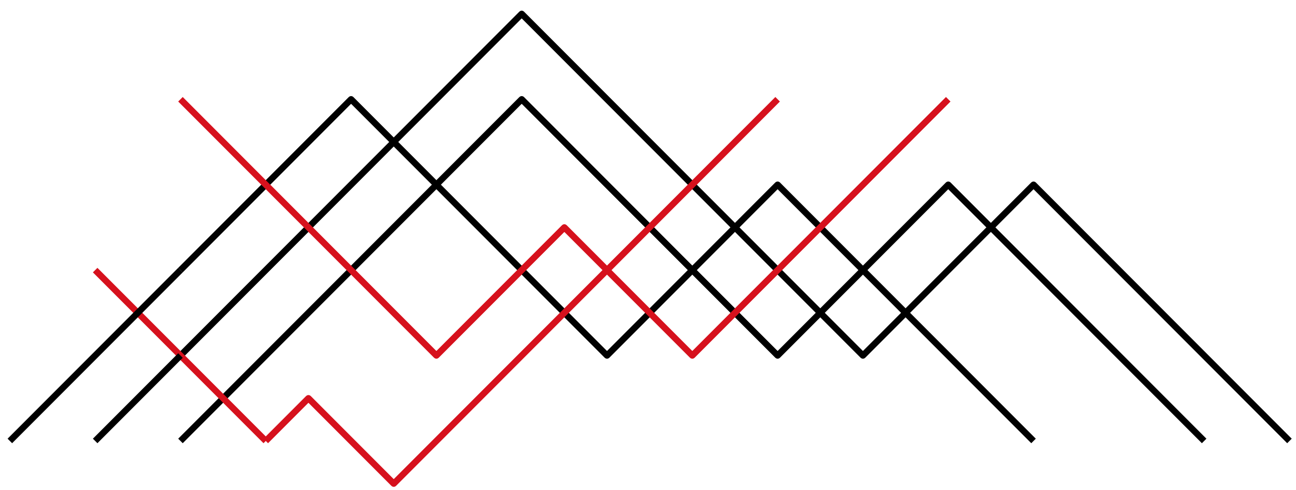

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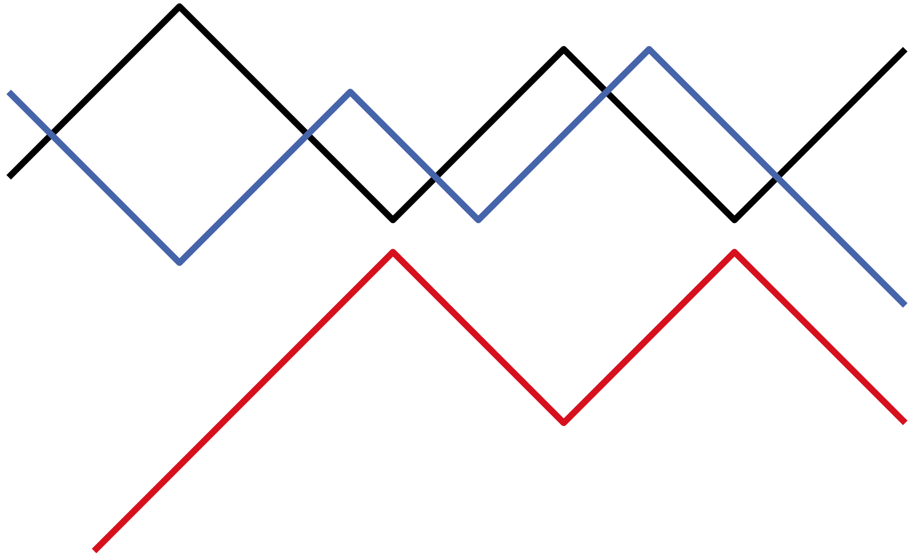

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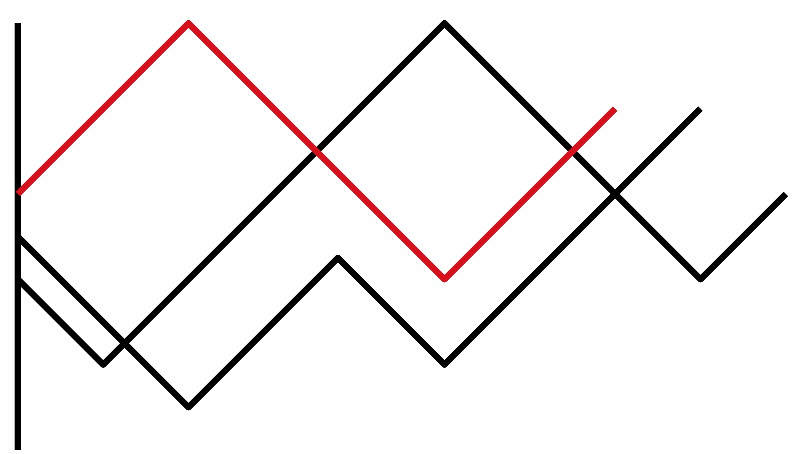

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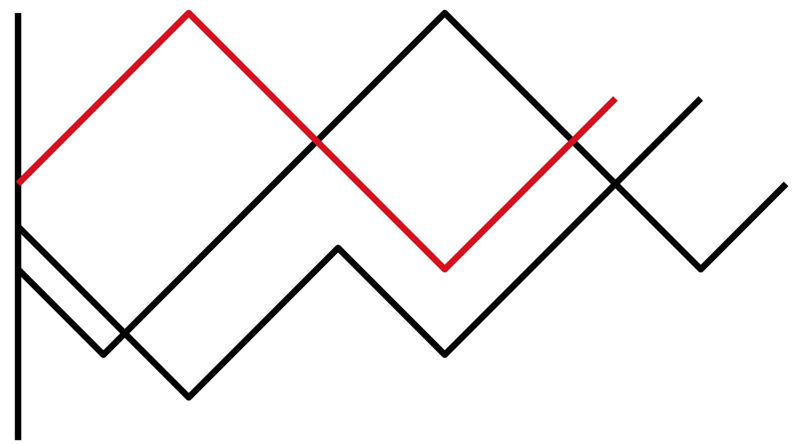
we can add a bend!



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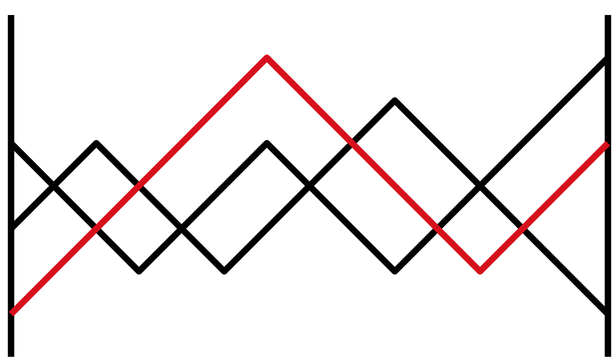

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Thm: For any $k \geq 0$, $\mathfrak{D}_k^{\nearrow} \subseteq \mathfrak{D}_k^{=} \subseteq \mathfrak{D}_k^{\leq} \subseteq \mathfrak{D}_k^{\vdash} \subseteq \mathfrak{D}_k^{\mathsf{H}} \subseteq \mathfrak{D}_{k+1}^{\nearrow}$.

Diagonal setting - main result

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Thm:

- For any odd $k \geq 1$, we have $\mathfrak{D}^{\mathsf{H}}_{k-1} \subsetneq \mathfrak{D}^{\nearrow}_{k} \subsetneq \mathfrak{D}^{=}_{k} = \mathfrak{D}^{\leq}_{k} = \mathfrak{D}^{\vdash}_{k} \subsetneq \mathfrak{D}^{\mathsf{H}}_{k}$.
- For any even $k \geq 2$, we have $\mathfrak{D}^{\mathsf{H}}_{k-1} \subsetneq \mathfrak{D}^{\nearrow}_{k} \subsetneq \mathfrak{D}^{=}_{k} \subsetneq \mathfrak{D}^{\leq}_{k} \subsetneq \mathfrak{D}^{\vdash}_{k} \subsetneq \mathfrak{D}^{\mathsf{H}}_{k}$.

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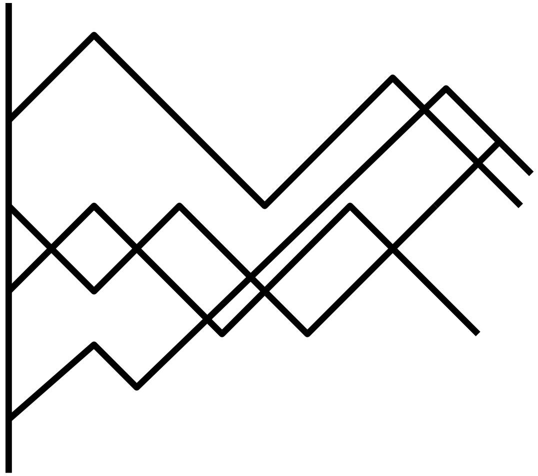

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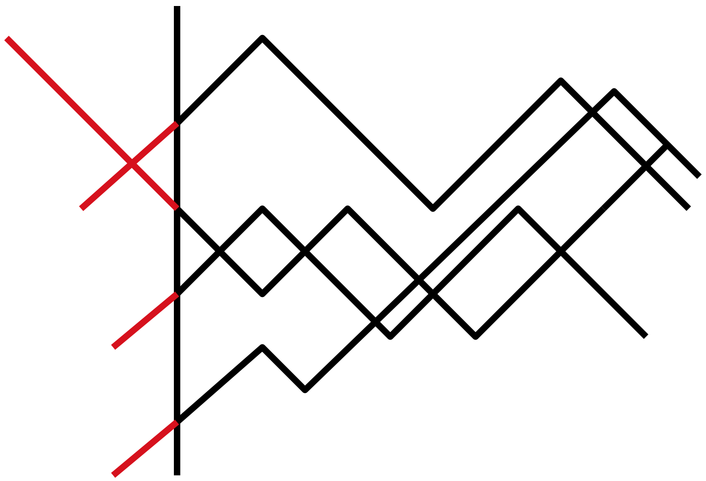

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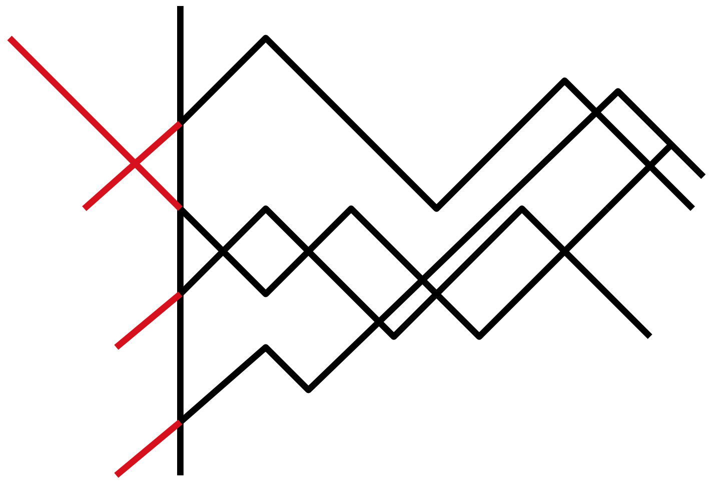

- New crossings to the left of the grounding line

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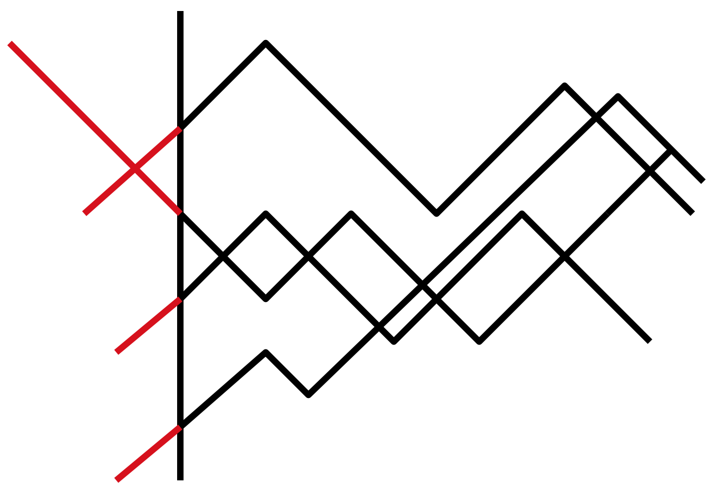

- New crossings to the left of the grounding line
- Extended curves cross $\Longleftrightarrow$ left slopes are different

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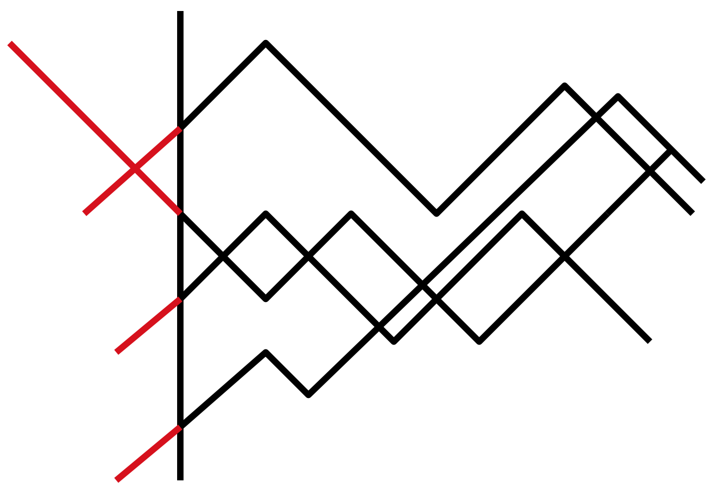

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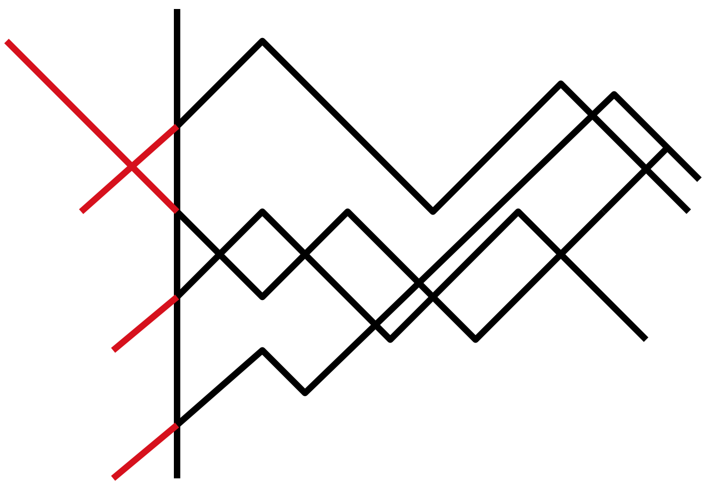

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- Extended curves cross $\iff$ left slopes are different
- Left slopes are different $\iff$ right slopes are different
- Extended curves cross $\iff$ original curves cross!

General setting (arbitrary slopes)

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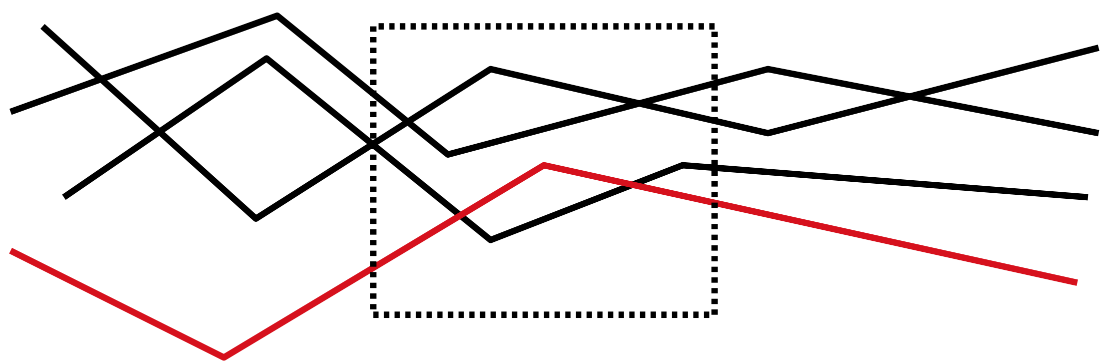

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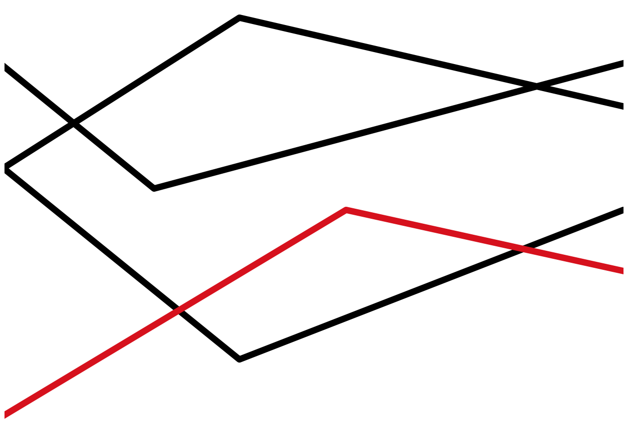

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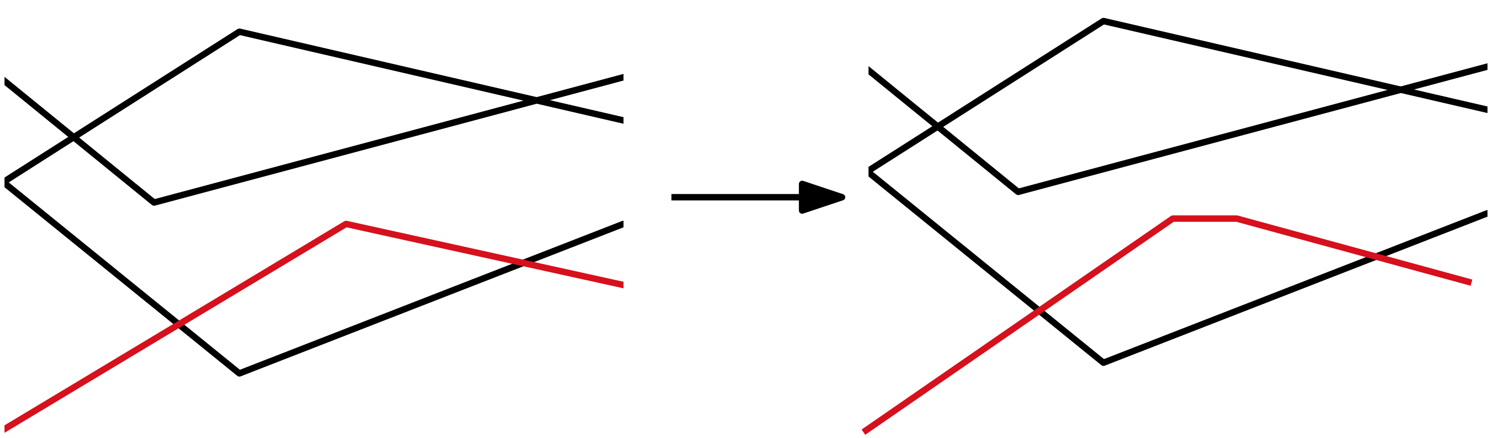

General setting (arbitrary slopes)

Price of *nice* representations = more things to distinguish

How many bounding lines do we have?

none = **unbounded**

one = **grounded**

two = **bounded**

Do we have exactly k bends? Doesn't matter!

Do all curves start with the same slope?

Not so natural to ask

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Three classes:

General setting (arbitrary slopes)

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Thm: For any k, $\mathfrak{G}_k^\mathsf{H} = \mathfrak{G}_k^\vdash$.

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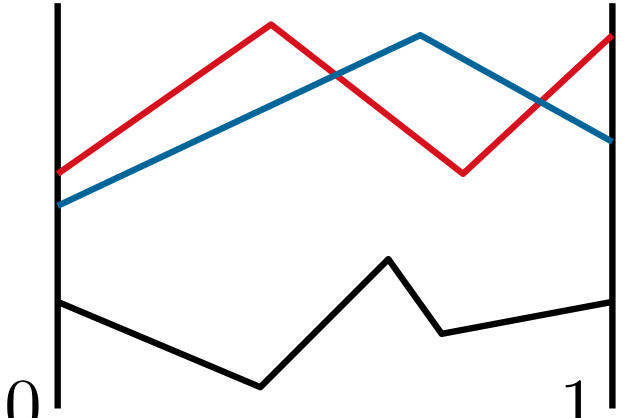

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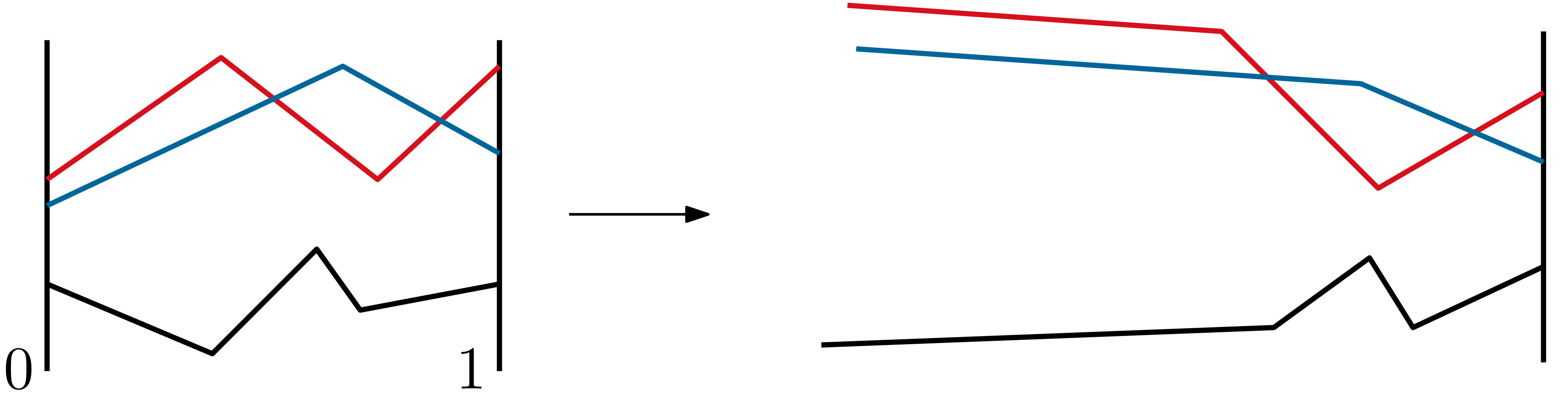

General setting - main result

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Thm: For any $k \geq 0$, $\mathfrak{G}_k^- \subseteq \mathfrak{G}_k^{\vdash} = \mathfrak{G}_k^{\mathsf{H}} \subsetneq \mathfrak{G}_{k+1}^-$.

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$\mathfrak{G}_k^{\mathsf{H}}$: bounded k bends

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Separation is witnessed by the following graph:

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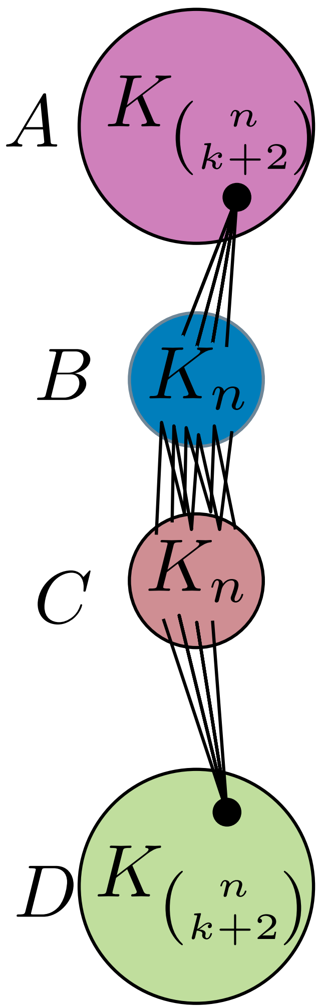

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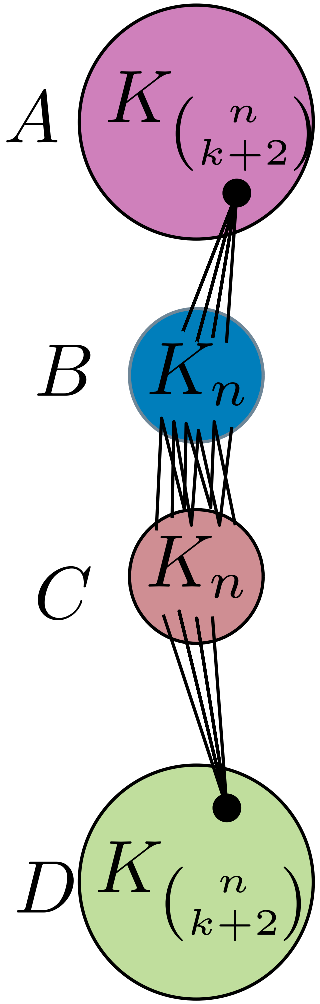

- n is very very large

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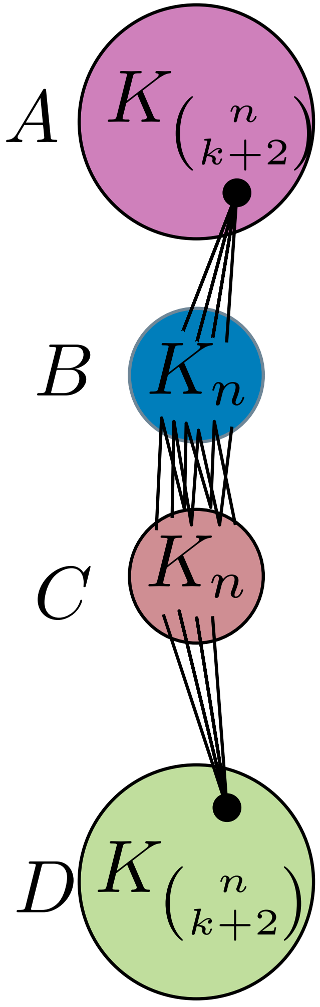

- n is very very large
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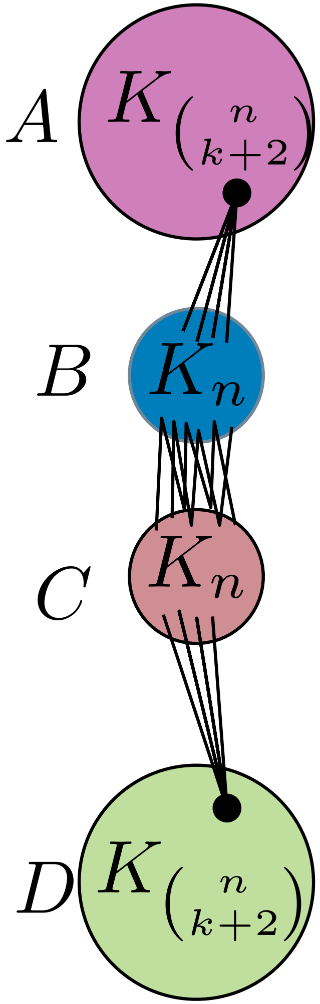

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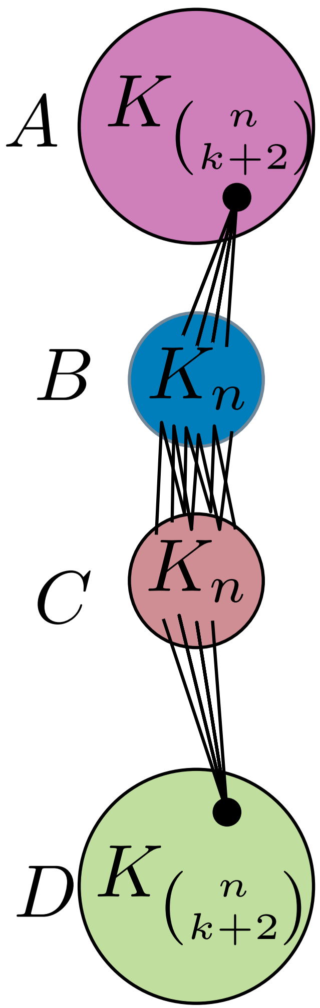

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- Every $(k+2)$-tuple in B is adjacent to a unique vertex in A
- Every $(k+2)$-tuple in C is adjacent to a unique vertex in D

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$\mathfrak{G}_{k+1}^{-}$: unbounded $(k+1)$ bends

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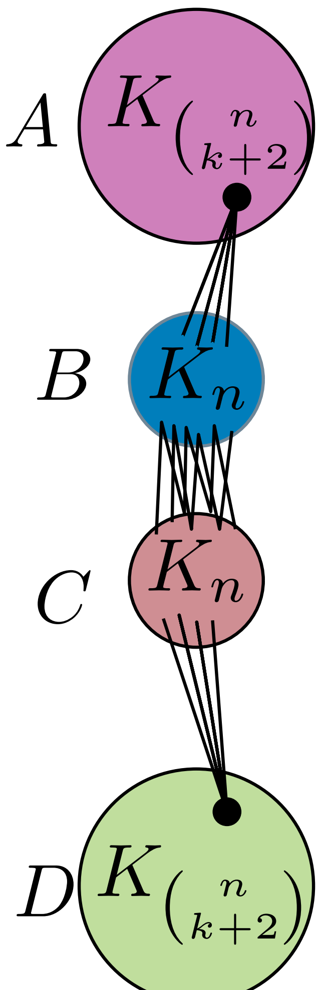

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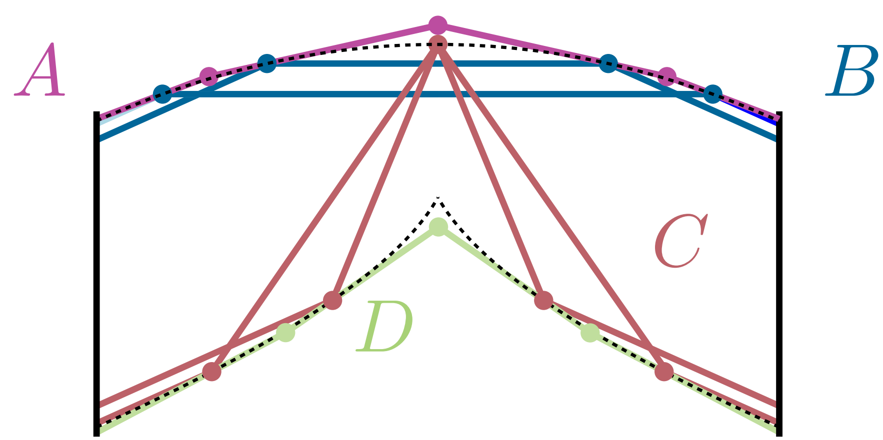

Thank you for attention!