On the structure of normalized models of circular-arc graphs
I. Hsu's approach

Tomasz Krawczyk

Warsaw University of Technology, Poland

The 33rd International Symposium
on Graph Drawing and Network Visualization
Norrköping, Sweden
September 24–26, 2025

Outline:

- geometric intersection graph classes: structure of their intersection models,
- modular decomposition tree – data structure representing intersection models of certain geometric intersection graphs,
- modular decomposition for circular-arc graphs:
  - PQSM-tree – data structure representing all normalized intersection models of circular-arc graphs,
  - inspired by the work of Wen-Lian Hsu from 1995,
    (this GD-work: Hsu's ideas and the errors in his work).

geometric intersection graphs

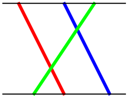
$\mathcal{S}$ – a family of geometric objects,

Intersection graphs of objects in $\mathcal{S}$ – graphs that admit an intersection model in $\mathcal{S}$:

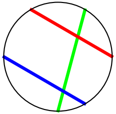
- the vertices represented by objects in $\mathcal{S}$ such that two vertices are adjacent $\iff$ the corresponding objects intersect.

Examples:

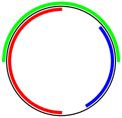
interval graphs



permutation graphs



circle graphs



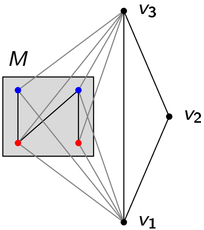
circular-arc graphs

modular decomposition tree

Modular decomposition tree – developed to represent all transitive orientations of comparability graphs,
Gallai, 1967

Modular decomposition tree of G represents the set of all modules of G.

module – a subset M of $V(G)$ such that every vertex from outside M is adjacent to every vertex in M or to no vertex in M.



modular decomposition tree $\mathcal{M}(G)$ of G:

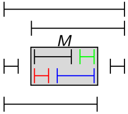
- consists of so-called strong modules of G, which ordered by $\subseteq$ form a tree,
- $V(G)$ is the root of $\mathcal{M}(G)$,
- $\{v\}$ for $v \in V(G)$ are the leaves of $\mathcal{M}(G)$.

strong modules in intersection models of interval/permutation graphs

Let M be a strong module in an interval/permutation graph G.

strong modules in intersection models of interval/permutation graphs

Let M be a strong module in an interval/permutation graph G.

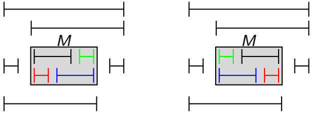


In every interval model of G:

- The smallest interval containing all intervals of M (gray box) is either disjoint from or contained in every other interval of the model.

strong modules in intersection models of interval/permutation graphs

Let M be a strong module in an interval/permutation graph G.



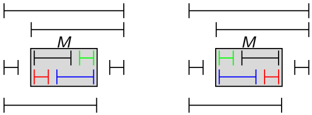
In every interval model of G:

- The smallest interval containing all intervals of M (gray box) is either disjoint from or contained in every other interval of the model.

We can reflect all intervals in M to obtain another model of G.

strong modules in intersection models of interval/permutation graphs

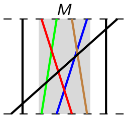
Let M be a strong module in an interval/permutation graph G.



In every interval model of G:

- The smallest interval containing all intervals of M (gray box) is either disjoint from or contained in every other interval of the model.

We can reflect all intervals in M to obtain another model of G.

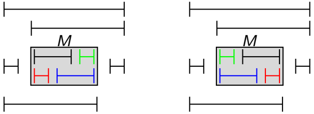


In every permutation model of G:

- The smallest box containing all segments of M (gray box) is either disjoint from every other segment of the model or crossed by it.

strong modules in intersection models of interval/permutation graphs

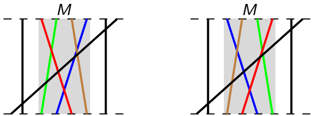
Let M be a strong module in an interval/permutation graph G.



In every interval model of G:

- The smallest interval containing all intervals of M (gray box) is either disjoint from or contained in every other interval of the model.

We can reflect all intervals in M to obtain another model of G.



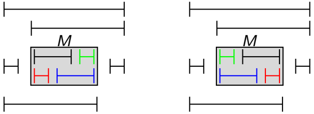
In every permutation model of G:

- The smallest box containing all segments of M (gray box) is either disjoint from every other segment of the model or crossed by it.

We can reflect all segments in M to obtain another model of G.

strong modules in intersection models of interval/permutation graphs

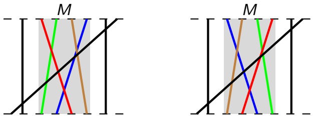
Let M be a strong module in an interval/permutation graph G.



In every interval model of G:

- The smallest interval containing all intervals of M (gray box) is either disjoint from or contained in every other interval of the model.

We can reflect all intervals in M to obtain another model of G.



In every permutation model of G:

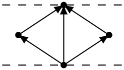
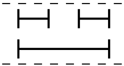
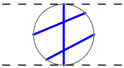
- The smallest box containing all segments of M (gray box) is either disjoint from every other segment of the model or crossed by it.

We can reflect all segments in M to obtain another model of G.

If G is prime (has no non-trivial modules),
then G has a unique (up to reflection and certain normalizations) model.

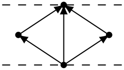
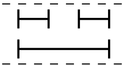
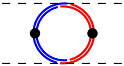
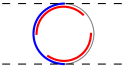
modular decomposition tree

Modular decomposition tree as a data structure representing models of some geometric intersection graphs.

	comparability graphs	modular decomp. tree	Gallai, 1967
	permutation graphs	modular decomp. tree	Gallai, 1967 Pnueli, Lempel, Even, 1971
	interval graphs	PQ –tree modular decomp. tree,	Booth and Lueker, 1976
	circle graphs	split decomp. tree	Cunningham, 1982

modular decomposition tree

Modular decomposition tree as a data structure representing models of some geometric intersection graphs.

	comparability graphs	modular decomp. tree	Gallai, 1967
	permutation graphs	modular decomp. tree	Gallai, 1967 Pnueli, Lempel, Even, 1971
	interval graphs	PQ –tree modular decomp. tree,	Booth and Lueker, 1976
	co-bipartite circular arc graphs	modular decomp. of the overlap graph	Spinrad, 1988
	circular-arc graphs	?	

normalized models of circular-arc graphs

normalized models of circular-arc graphs

G – a circular-arc graph, no twins and no universal vertices,

normalized models of circular-arc graphs

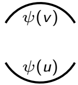
G – a circular-arc graph, no twins and no universal vertices,

normalized models of G: relative relations between the arcs reflect relations between the closed neighborhoods of the vertices of G.

normalized models of circular-arc graphs

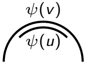
G – a circular-arc graph, no twins and no universal vertices,

normalized models of G: relative relations between the arcs reflect relations between the closed neighborhoods of the vertices of G.



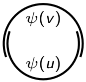
$uv \notin E(G)$

u and v are disjoint in G



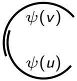
$N[u] \subsetneq N[v]$

u is contained in v in G



$$
\begin{gathered}
uv \in E(G), \\
N[u] \cup N[v] = V(G), \\
w \in N[u] \setminus N[v] \implies N[w] \subsetneq N[u], \\
w \in N[v] \setminus N[u] \implies N[w] \subsetneq N[v]
\end{gathered}
$$

u and v cover the circle in G



otherwise

u and v overlap in G, denoted $u \sim v$.

G_{ov} – the overlap graph of G.

normalized models $\Longrightarrow$ chord models (Spinrad 1998, Hsu 1996)

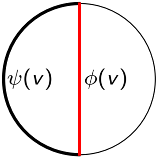
- ψ – a normalized model of G,

normalized models $\Longrightarrow$ chord models (Spinrad 1998, Hsu 1996)

- ψ – a normalized model of G,
- convert every arc $\psi(v)$ into chord $\phi(v)$ with the same endpoints

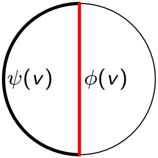
normalized models $\implies$ chord models (Spinrad 1998, Hsu 1996)

- ψ – a normalized model of G,
- convert every arc $\psi(v)$ into chord $\phi(v)$ with the same endpoints

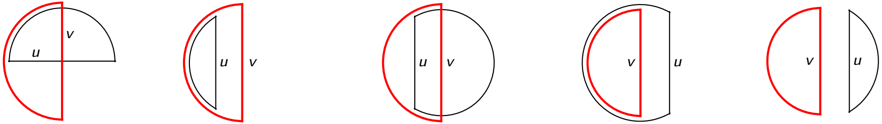


normalized models $\Longrightarrow$ chord models (Spinrad 1998, Hsu 1996)

- ψ – a normalized model of G,
- convert every arc $\psi(v)$ into chord $\phi(v)$ with the same endpoints



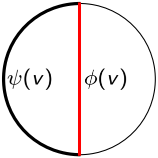
Pairs of vertices of G:



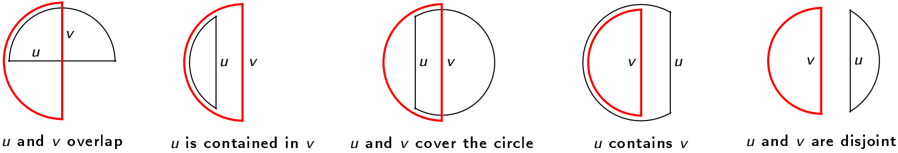
u and v overlap | u is contained in v | u and v cover the circle | u contains v | u and v are disjoint

normalized models $\Longrightarrow$ chord models (Spinrad 1998, Hsu 1996)

- ψ – a normalized model of G,
- convert every arc $\psi(v)$ into chord $\phi(v)$ with the same endpoints



Pairs of vertices of G:



normalized models of G $\Longrightarrow$ chord models of G_{ov} (G_{ov} is a circle graph!)

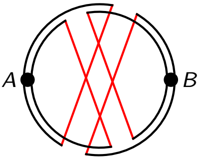
co-bipartite circular-arc graphs, Spinrad 1988

$G = (A, B, E)$ – a co-bipartite circular-arc graphs,
vertex set is partitioned into two cliques: A and B,

co-bipartite circular-arc graphs, Spinrad 1988

$G = (A, B, E)$ – a co-bipartite circular-arc graphs,
vertex set is partitioned into two cliques: A and B,

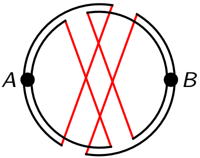
- normalized model of $G \implies$ strongly normalized models of G:
  - arcs for the vertices of A contain the leftmost point of the circle,
  - arcs for the vertices of B contain the rightmost point of the circle,
- strongly normalized model of $G \implies$ chord models of G_{ov}.



co-bipartite circular-arc graphs, Spinrad 1988

$G = (A, B, E)$ – a co-bipartite circular-arc graphs,
vertex set is partitioned into two cliques: A and B,

- normalized model of $G \implies$ strongly normalized models of G:
  - arcs for the vertices of A contain the leftmost point of the circle,
  - arcs for the vertices of B contain the rightmost point of the circle,
- strongly normalized model of $G \implies$ chord models of G_{ov}.

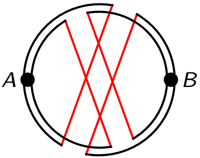


- G_{ov} – permutation graph!
- modular decomposition tree of G_{ov} represents all s.n. models of G,

co-bipartite circular-arc graphs, Spinrad 1988

$G = (A, B, E)$ – a co-bipartite circular-arc graphs,
vertex set is partitioned into two cliques: A and B,

- normalized model of $G \implies$ strongly normalized models of G:
  - arcs for the vertices of A contain the leftmost point of the circle,
  - arcs for the vertices of B contain the rightmost point of the circle,
- strongly normalized model of $G \implies$ chord models of G_{ov}.

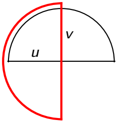


- G_{ov} – permutation graph!
- modular decomposition tree of G_{ov} represents all s.n. models of G,
- G_{ov} - prime $\implies$ G has a unique s. n. model.

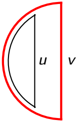
Hsu's approach

An attempt to generalize Spinrad's work to the class of all circular-arc graphs:

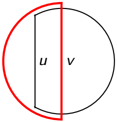
- normalized models of $G \implies$ chord models of G_{ov},
- G_{ov} is a circle graph (we can not use Gallai's tools).



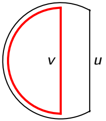
u and v overlap



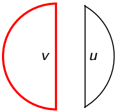
u is contained in v



u and v cover the circle



u contains v

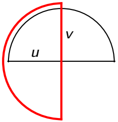


u and v are disjoint

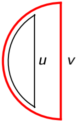
Hsu's approach

An attempt to generalize Spinrad's work to the class of all circular-arc graphs:

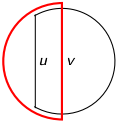
- normalized models of G $\implies$ chord models of G_{ov},
- G_{ov} is a circle graph (we can not use Gallai's tools).



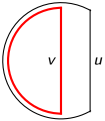
u and v overlap



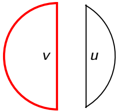
u is contained in v



u and v cover the circle



u contains v



u and v are disjoint

conformal models of G_{ov} – chord models of G_{ov} corresponding to normalized models of G.

Hsu's approach – the first idea

If G_{ov} is prime, then G_{ov} admits a unique conformal model.

Proof idea (induction):

- decompose G_{ov} into smaller prime graphs,
- argue that their unique conformal models can be uniquely combined into a conformal model of G_{ov}.

Hsu's approach – the first idea

If G_{ov} is prime, then G_{ov} admits a unique conformal model.

Proof idea (induction):

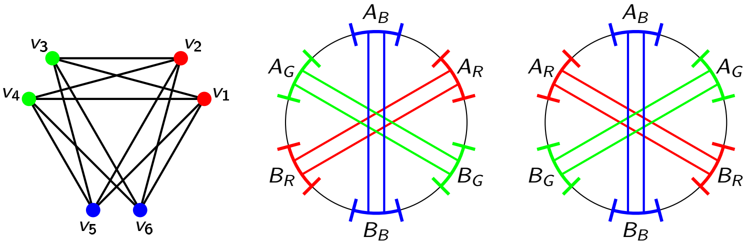
- decompose G_{ov} into smaller prime graphs,
- argue that their unique conformal models can be uniquely combined into a conformal model of G_{ov}.

The smaller graphs determined by Hsu are not prime (the counterexample in the paper).

Hsu's approach – the second idea

The vertex set $V(G)$ can be partitioned into so-called consistent modules, intended to satisfy the following properties:

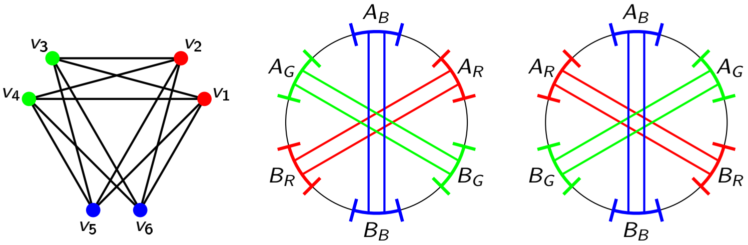
- For every conformal model of G_{ov} and every consistent module R, the chords representing the vertices in R are spanned between two arcs A_R and B_R, which contain no endpoint of any chord corresponding to a vertex outside R.
- The possible placements of the arcs A_R and B_R in conformal models of G_{ov} can be represented by the modular decomposition of G_{ov}.



Hsu's approach – the second idea

The vertex set $V(G)$ can be partitioned into so-called consistent modules, intended to satisfy the following properties:

- For every conformal model of G_{ov} and every consistent module R, the chords representing the vertices in R are spanned between two arcs A_R and B_R, which contain no endpoint of any chord corresponding to a vertex outside R.
- The possible placements of the arcs A_R and B_R in conformal models of G_{ov} can be represented by the modular decomposition of G_{ov}.



The consistent modules defined by Hsu do not satisfy the properties listed above (the counterexample in the paper).

Hsu's work

In his work (1) from 1995, Wen-Lian Hsu claimed three results:

- the design of so-called decomposition trees representing the structure of all normalized intersection models of circular-arc graphs,
- an $\mathcal{O}(mn)$-time recognition algorithm for circular-arc graphs,
- an $\mathcal{O}(mn)$-time isomorphism algorithm for circular-arc graphs.

Curtis, Lin, McConnell, Nussbaum, Soulignac, Spinrad, Szwarcfiter (2013):

- reported an error in Hsu's isomorphism algorithm.

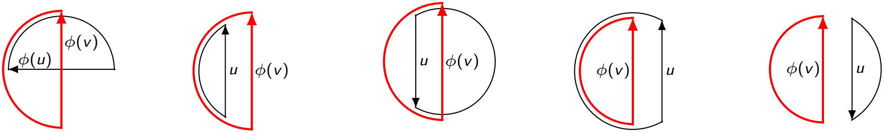
T.K. (2025):

- the other two results of Hsu's work are also incorrect,
  Hsu's decomposition trees are not constructed properly,
- The Gallai-Spinrad-Hsu framework can be made to work! (2020)

(1) Wen-Lian Hsu, *$\mathcal{O}(mn)$-time algorithms for the recognition and isomorphism problems on circular-arc graphs,*
SIAM J. Comput. 24(3), 411–439, 1995.

our approach

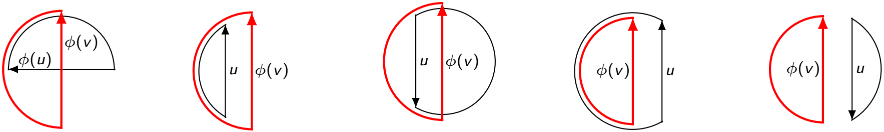
We orient the chord $\phi(v)$ associated with the arc $\psi(v)$ so that the arc $\phi(v)$ is on the left side of the oriented chord $\phi(v)$.



u and v overlap | u is contained in v | u and v cover the circle | u contains v | u and v are disjoint

our approach

We orient the chord $\phi(v)$ associated with the arc $\psi(v)$ so that the arc $\phi(v)$ is on the left side of the oriented chord $\phi(v)$.



u and v overlap | u is contained in v | u and v cover the circle | u contains v | u and v are disjoint

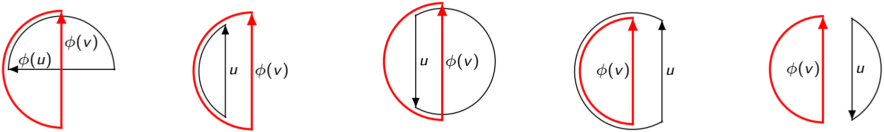
$$\begin{aligned} left(v) &= \{u \in V(G) : u \text{ is contained in } v \text{ or } u \text{ and } v \text{ cover the circle}\}, \\ right(v) &= \{u \in V(G) : u \text{ contains } v \text{ or } u \text{ and } v \text{ are disjoint}\}. \end{aligned}$$

Conformal models of G_{ov} – oriented chord models of G_{ov} such that for every $v \in V(G)$:

- the chords for vertices in $left(v)$ are on the left side of the chord for v,
- the chords for vertices in $right(v)$ are on the right side of the chord for v.

our approach

We orient the chord $\phi(v)$ associated with the arc $\psi(v)$ so that the arc $\phi(v)$ is on the left side of the oriented chord $\phi(v)$.



u and v overlap — u is contained in v — u and v cover the circle — u contains v — u and v are disjoint

$$
\begin{aligned}
\textit{left}(v) &= \{u \in V(G) : u \text{ is contained in } v \text{ or } u \text{ and } v \text{ cover the circle}\},\\
\textit{right}(v) &= \{u \in V(G) : u \text{ contains } v \text{ or } u \text{ and } v \text{ are disjoint}\}.
\end{aligned}
$$

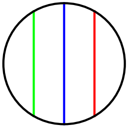
Conformal models of G_{ov} – oriented chord models of G_{ov} such that for every $v \in V(G)$:

- the chords for vertices in $\textit{left}(v)$ are on the left side of the chord for v,
- the chords for vertices in $\textit{right}(v)$ are on the right side of the chord for v.

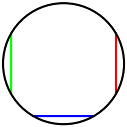
$$\text{normalized models of } G \iff \text{conformal models of } G_{ov}$$

two approaches - a comparison

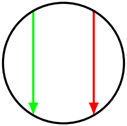
- Hsu tries to describe how to transform between conformal models so as to keep parallel/serial relation between non-intersecting triples of chords unchanged,
- We try to describe how to transform between conformal models that keep left/right relation between every two non-intersecting chords unchanged.



parallel triple

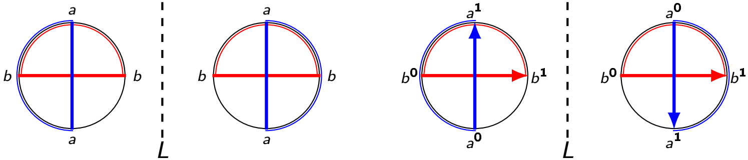


serial triple



two approaches - a comparison

- The reflection of two overlapping arcs *a* and *b* yields two intersecting chords, (no effect is seen).
- The reflection of two overlapping arcs *a* and *b* yields two intersecting orineted chords, with the head of *a* switching from the left side of *b* to its right side.



results

T.K. (2020):
PQSM-tree (refined modular decomposition of the overlap graph) – a data structure representing all normalized models of circular-arc graphs,
(can be constructed in linear-time.)

results

T.K. (2020):
PQSM-tree (refined modular decomposition of the overlap graph) – a data structure representing all normalized models of circular-arc graphs,
(can be constructed in linear-time.)

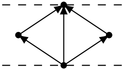
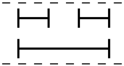
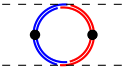
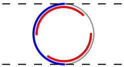
T.K. (2020):
The isomorphism and canonization problem in the class of circular-arc graphs can be solved in linear-time.

Remarks:

- Our proof havily uses the ideas introduced by Hsu.

modular decomposition tree

Modular decomposition tree as a data structure representing models of some geometric intersection graphs.

	comparability graphs	modular decomp. tree	Gallai, 1967
	permutation graphs	modular decomp. tree	Gallai, 1967 Pnueli, Lempel, Even, 1971
	interval graphs	PQ –tree modular decomp. tree,	Booth and Lueker, 1976
	co-bipartite circular arc graphs	modular decomp. of the overlap graph	Spinrad, 1988
	circular-arc graphs	modular decomp. of the overlap graph	T.K. (2020), inspired by Hsu (1995).

Thank you.