Treewidth of Outer k-Planar Graphs

Rafał Pyzik

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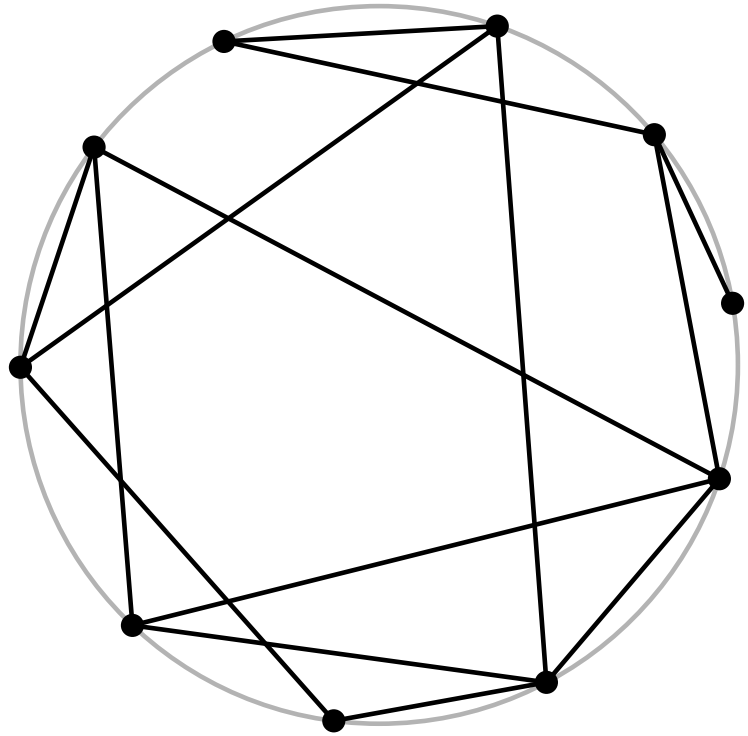
Krakow, Poland

Outer (min-)k-planar graphs

Convex drawing:

- straight-line
- vertices on a circle

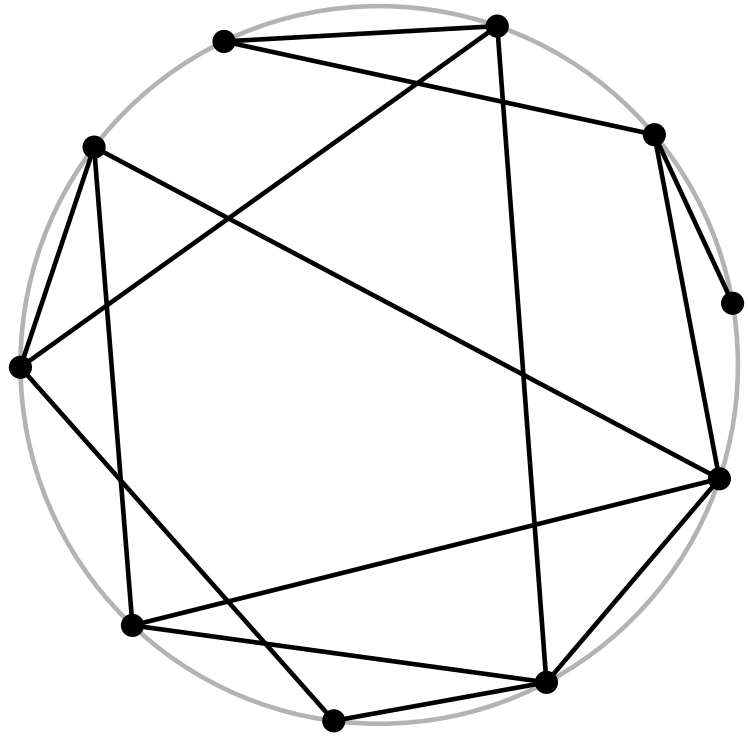


Outer (min-)k-planar graphs

Outer k-planar graph

admits a drawing:

- straight-line
- vertices on a circle
- k-planar



outer 3-planar graph

Outer (min-)k-planar graphs

Outer min-k-planar graph

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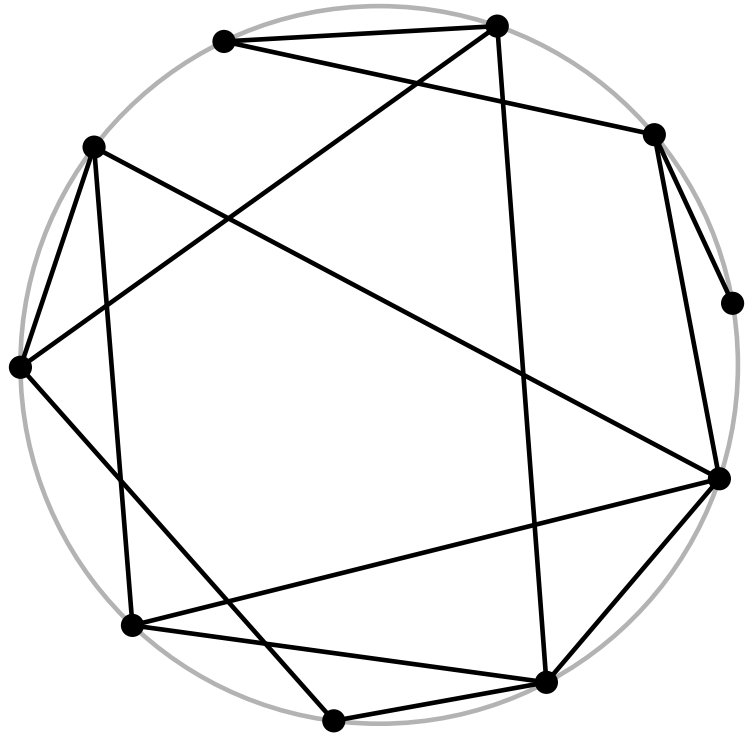
- vertices on a circle
- min-k-planar



outer min-2-planar graph

Outer (min-)k-planar graphs

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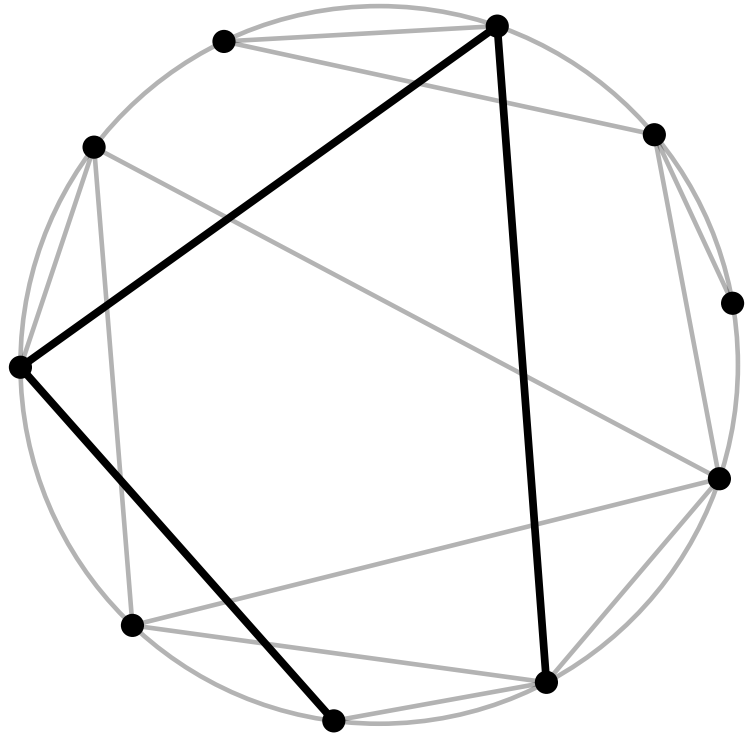
- vertices on a circle
- min-k-planar



outer min-2-planar graph

Previous bounds and new results

Treewidth:

	upper	lower
outer k-planar	$1.5k + 2$	$k + 2$
outer min-k-planar	$3k + 1$	$k + 2$

All previous bounds: Firman, Gutowski, Kryven, Okada, Wolff [GD 2024]

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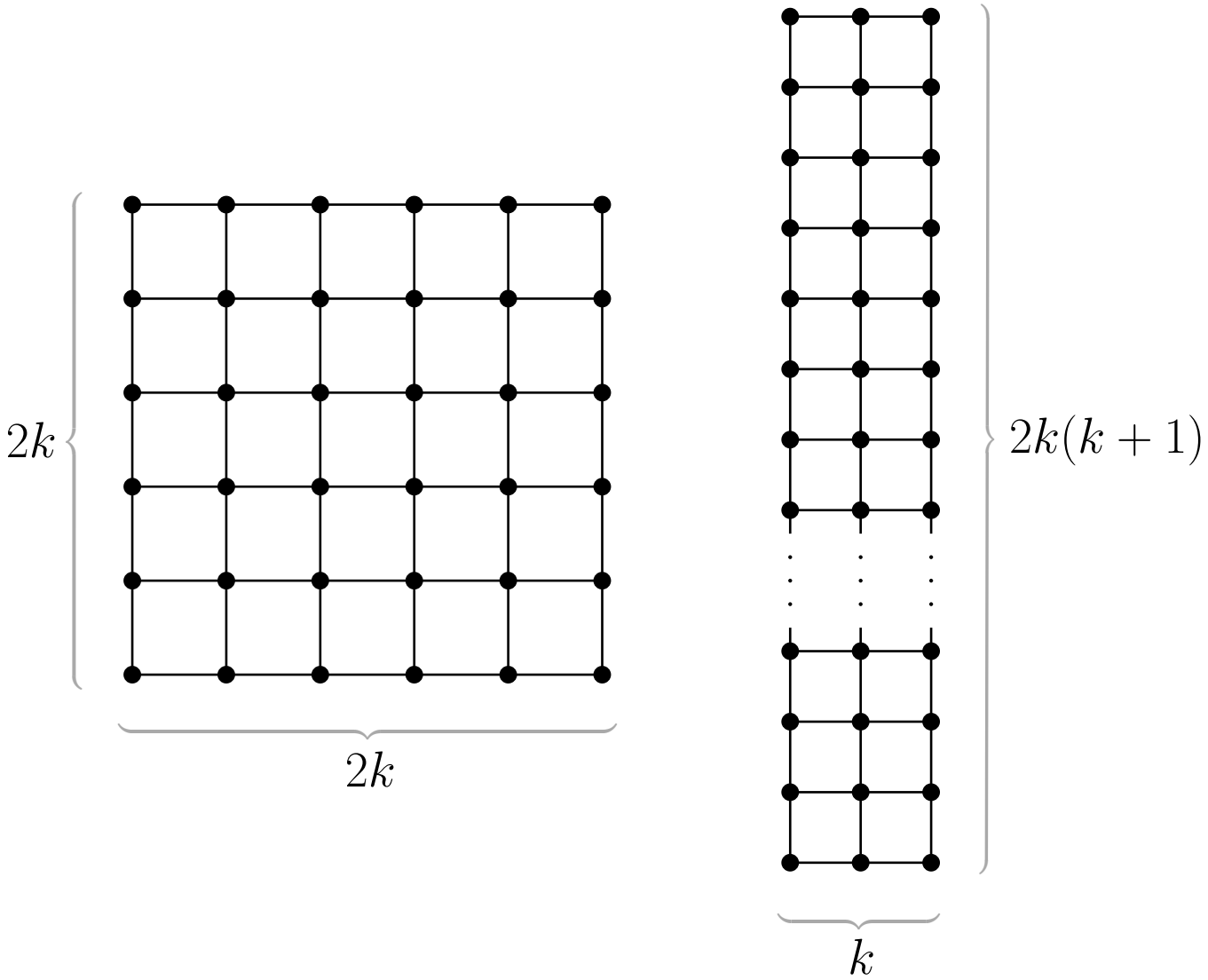
Construction for the lower bound



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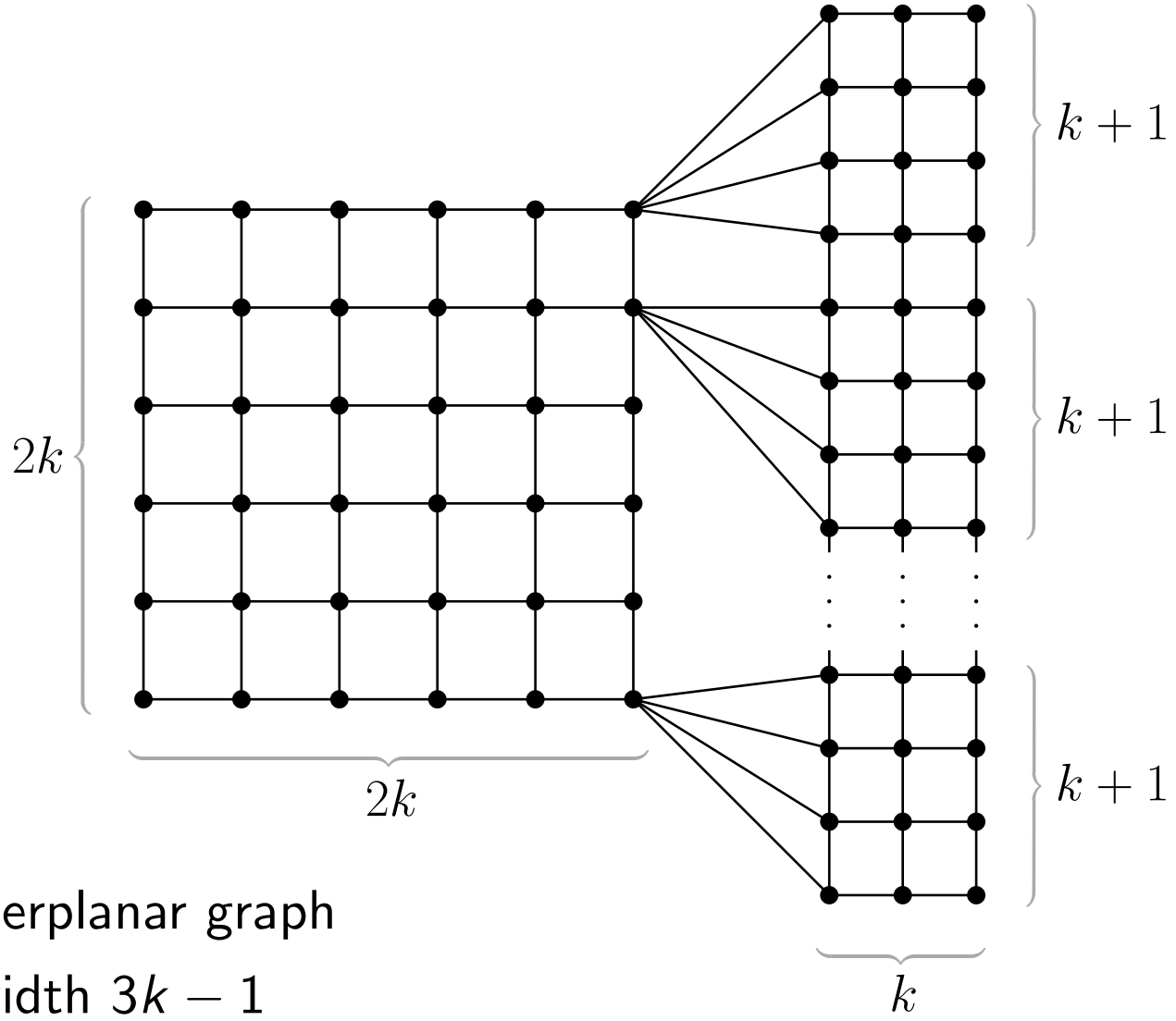

- k-outerplanar graph
- treewidth $3k-1$
- appeared before (Kammer, Tholey)

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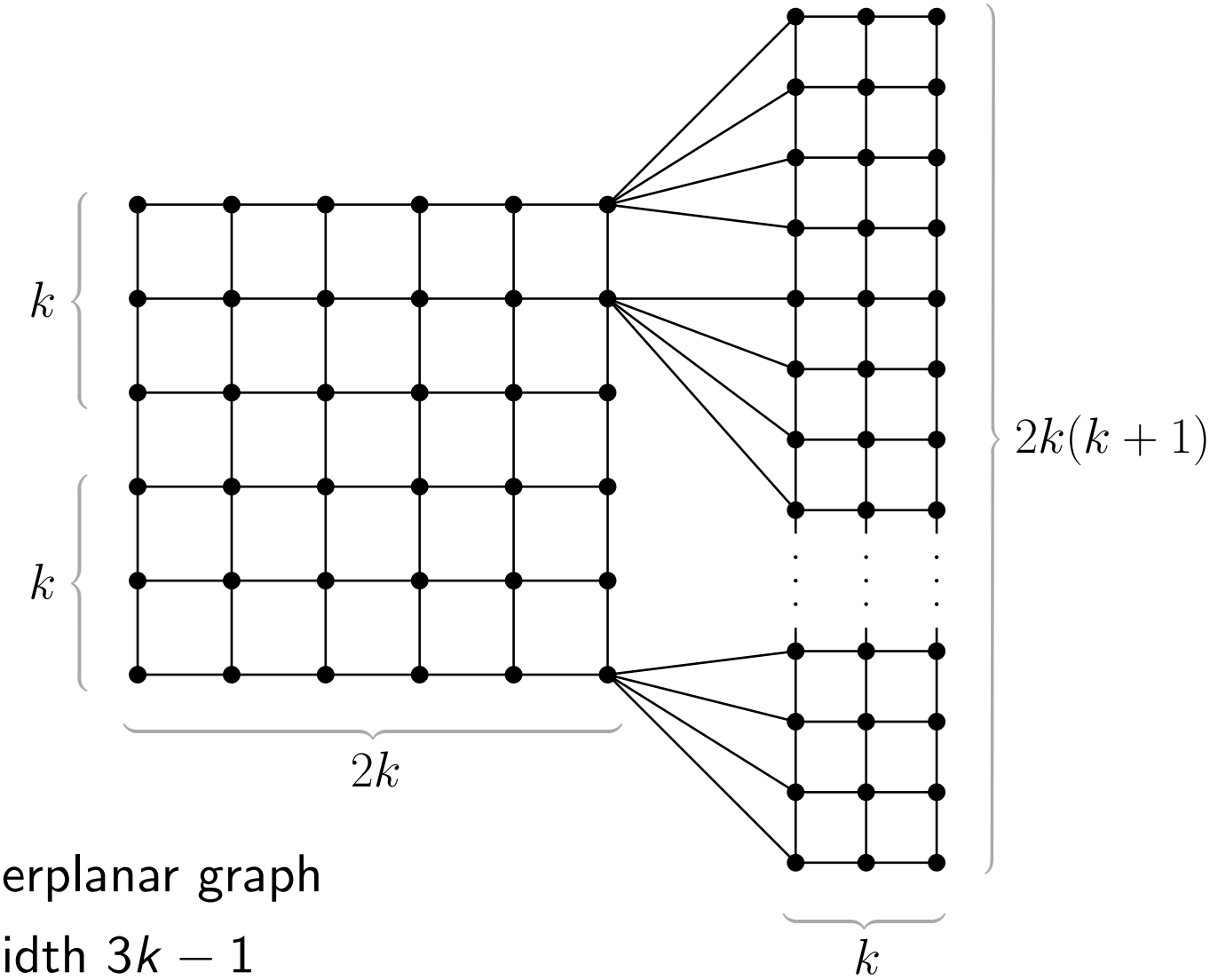

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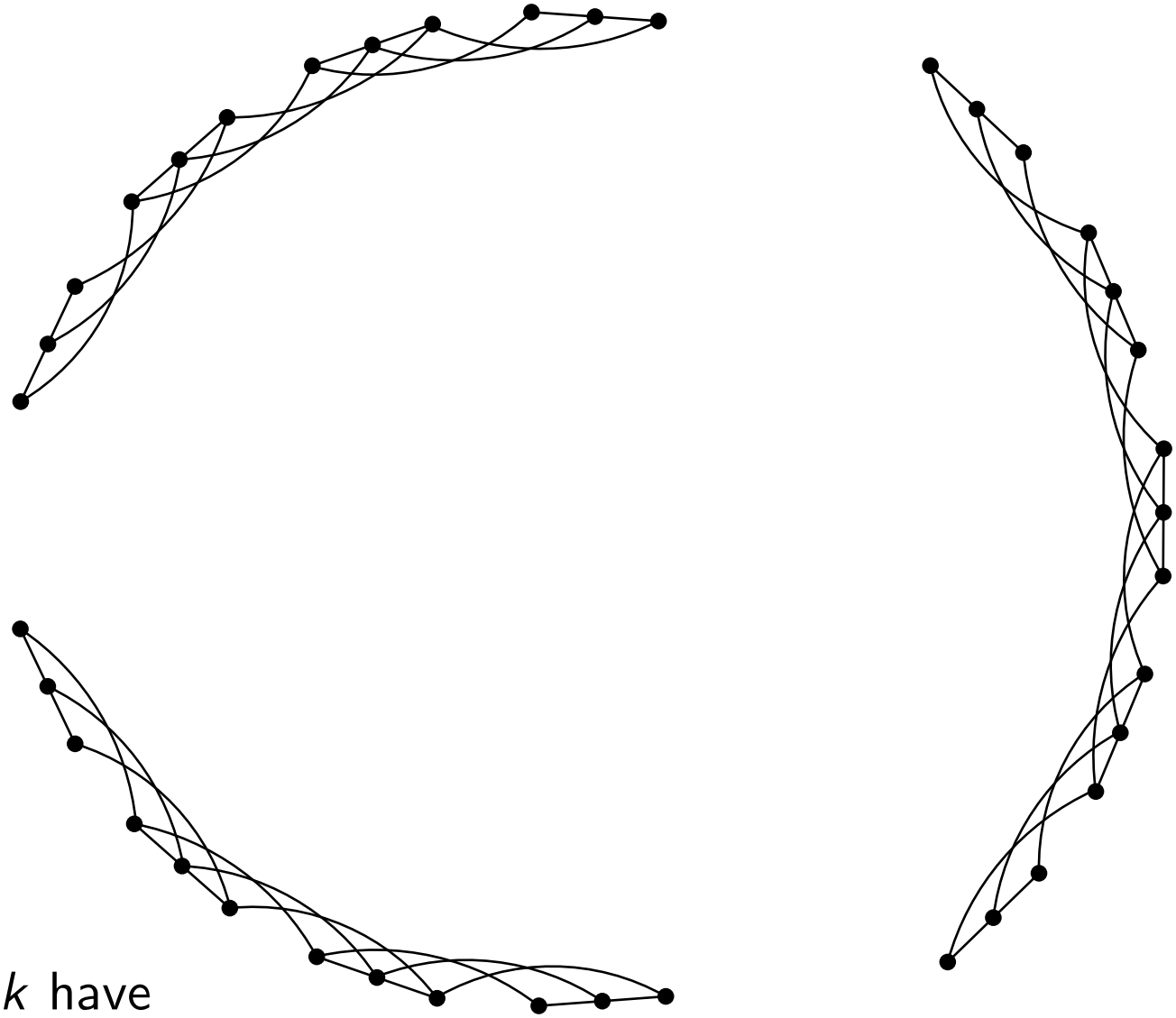

Grids of width k have
outer $(2k-2)$-planar drawing.

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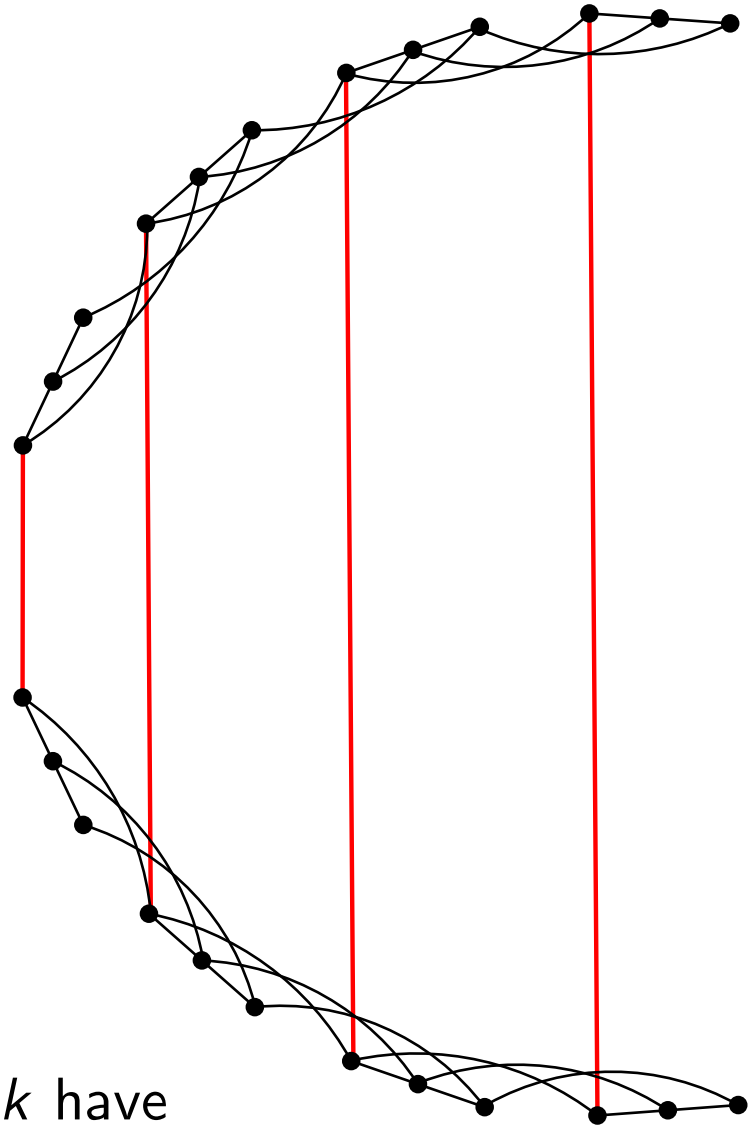

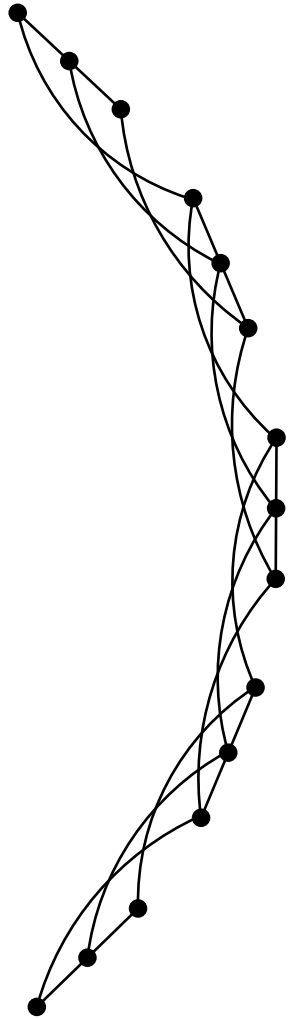

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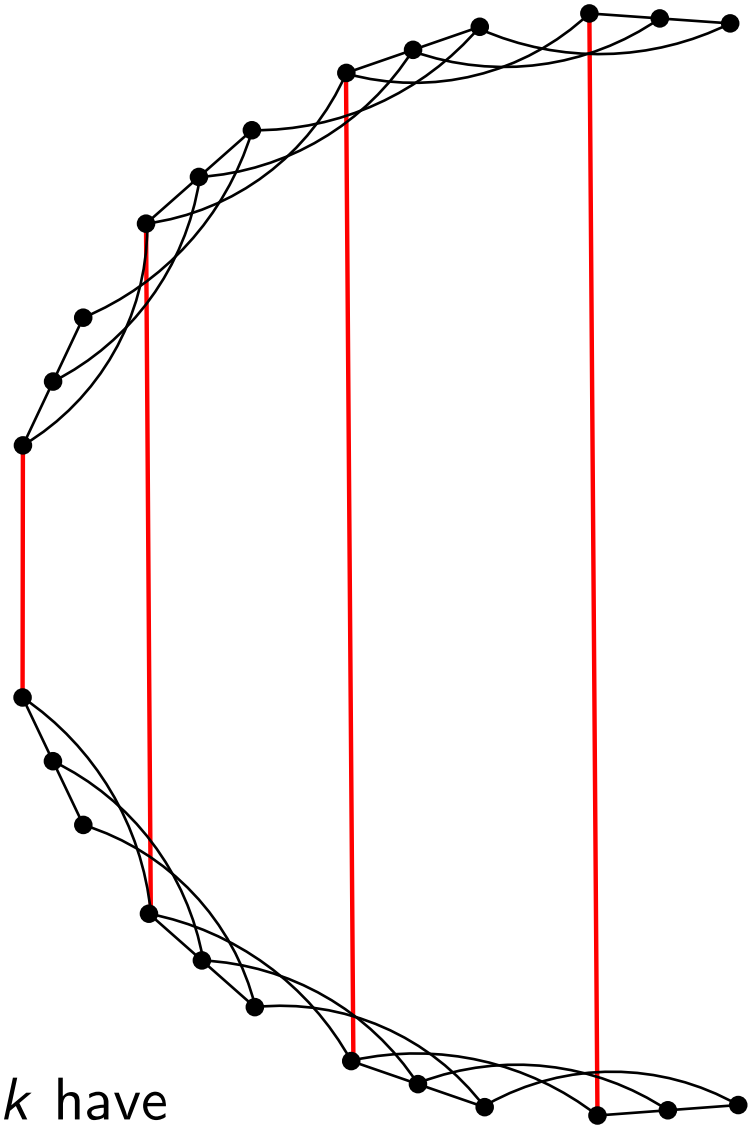

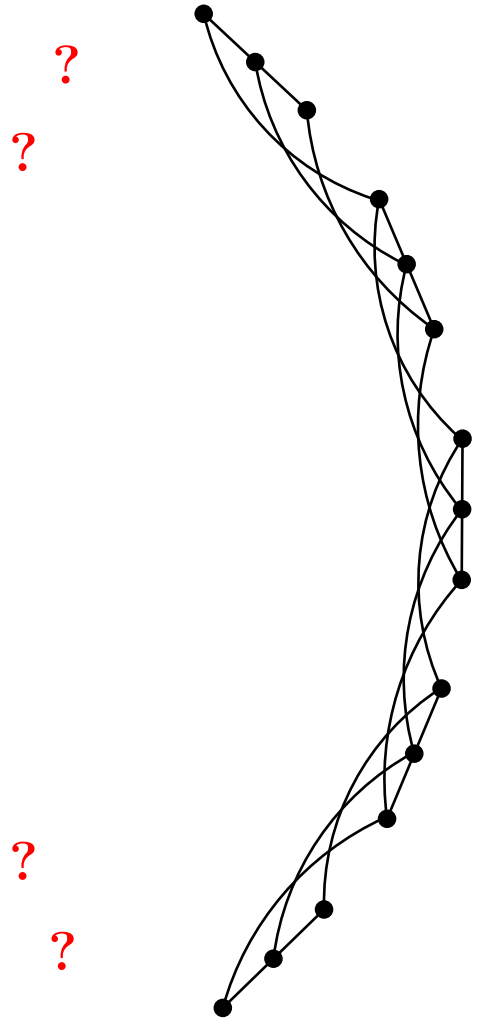

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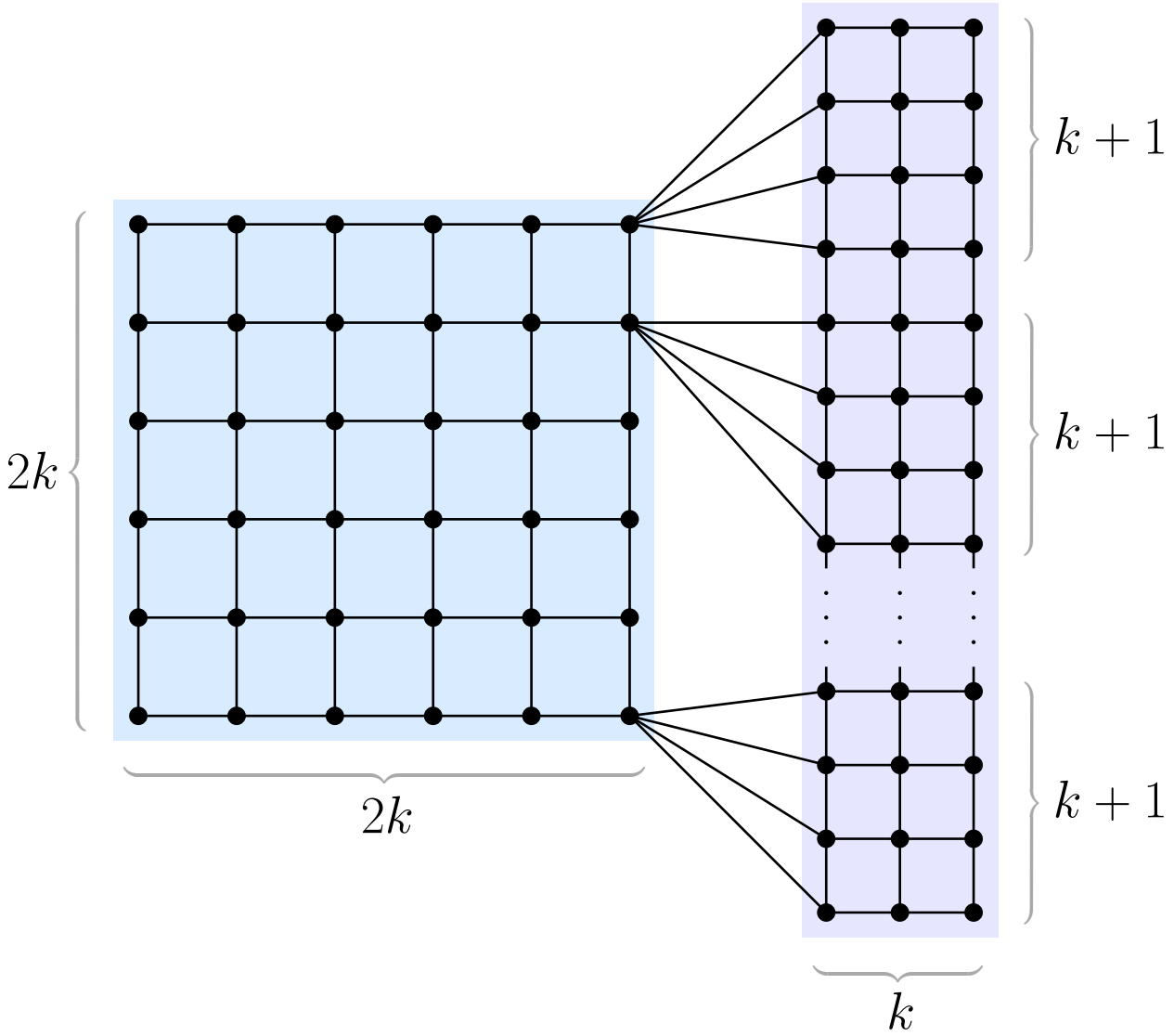

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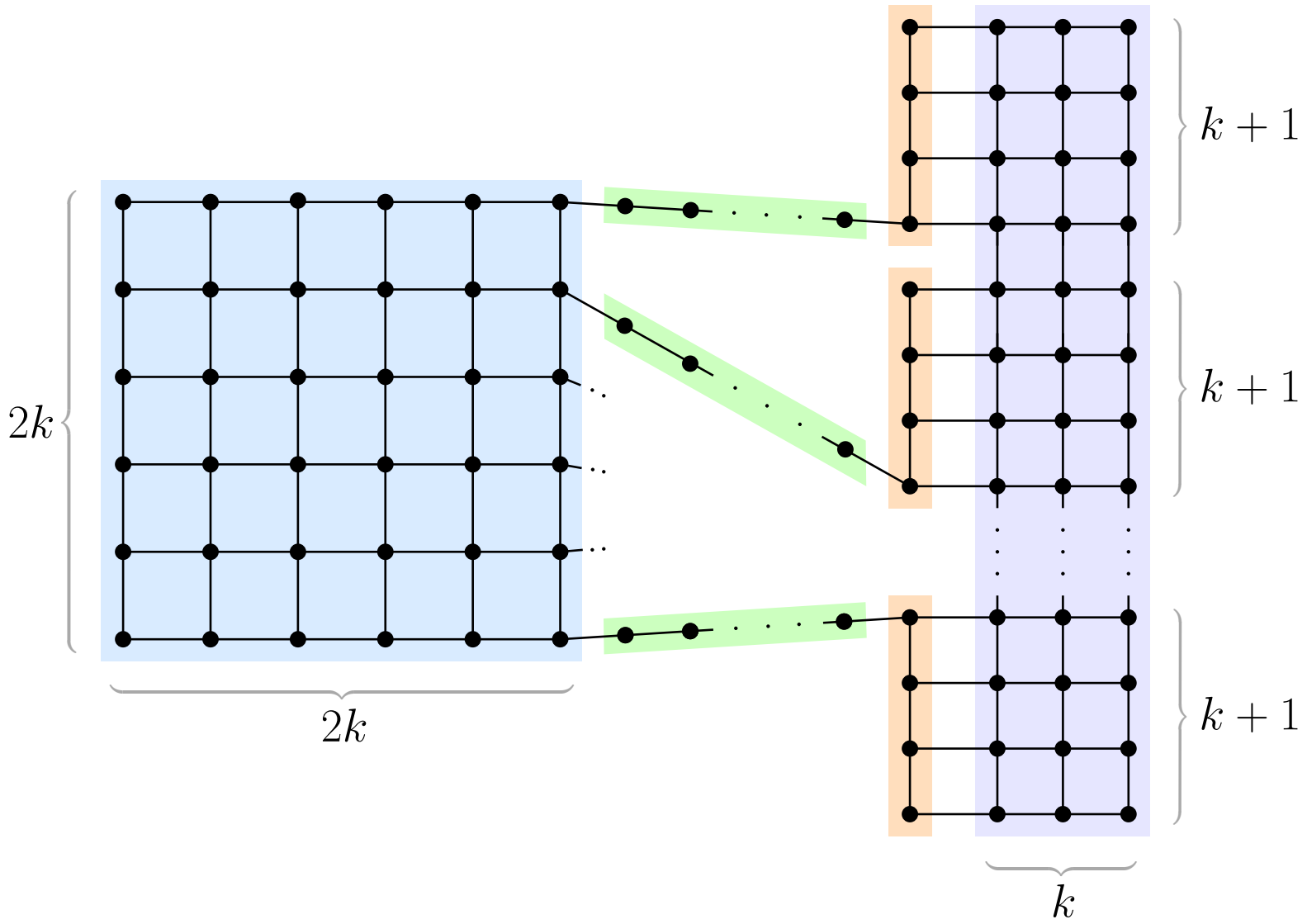
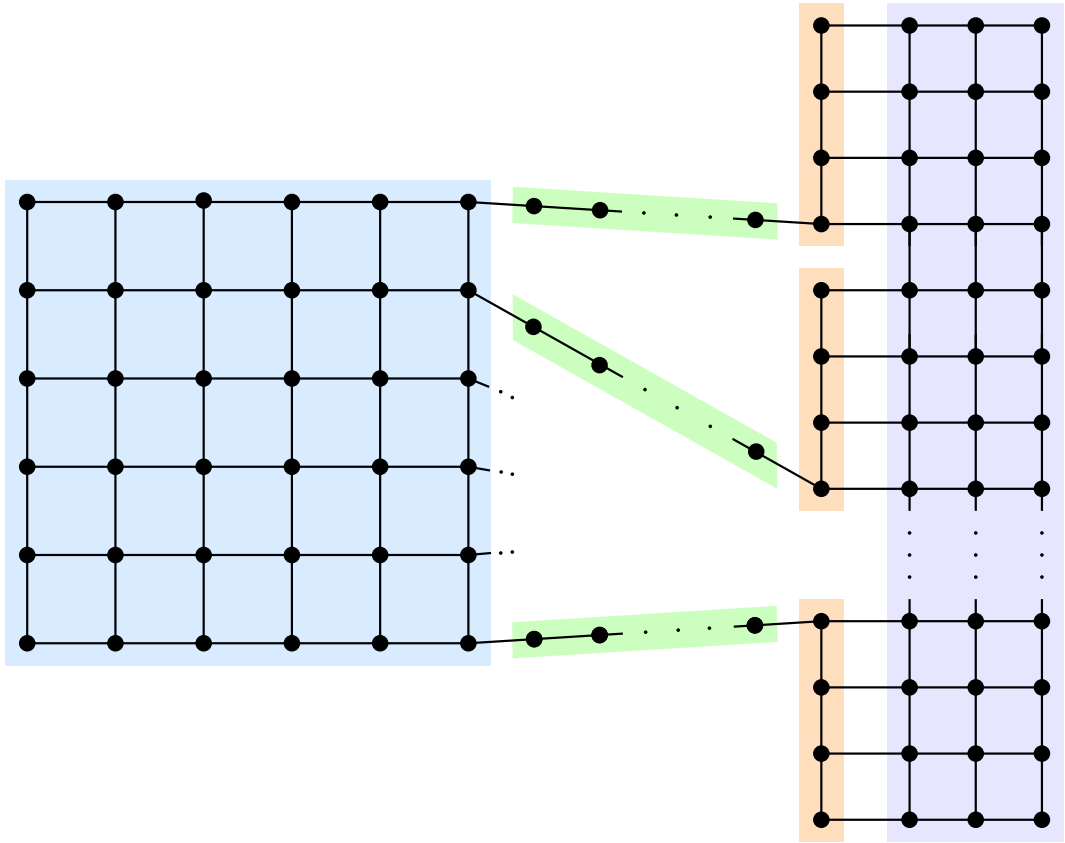


Construction for the lower bound

Green paths are sufficiently long.



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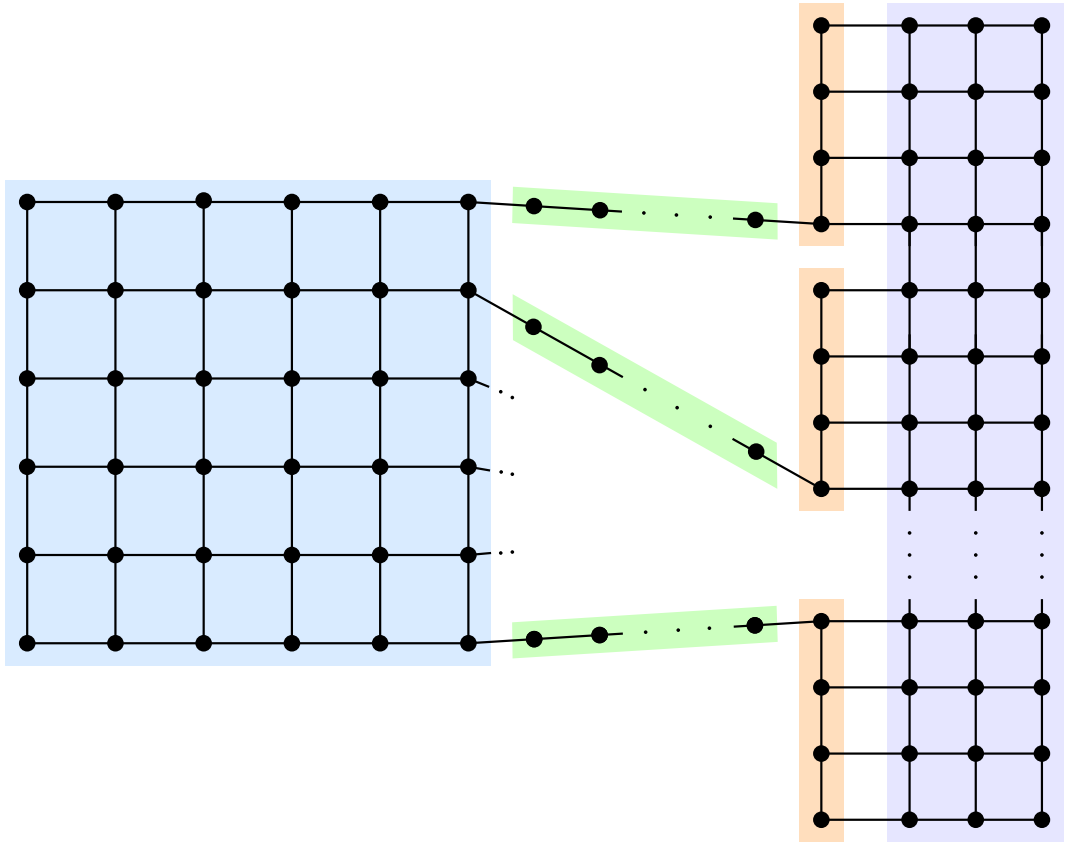
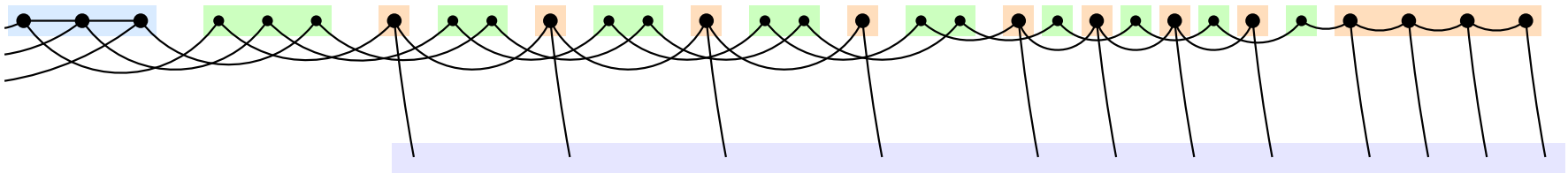


Orange vertices are separated by green paths.



Upper bounds for outer min-k-planar graphs

Theorem

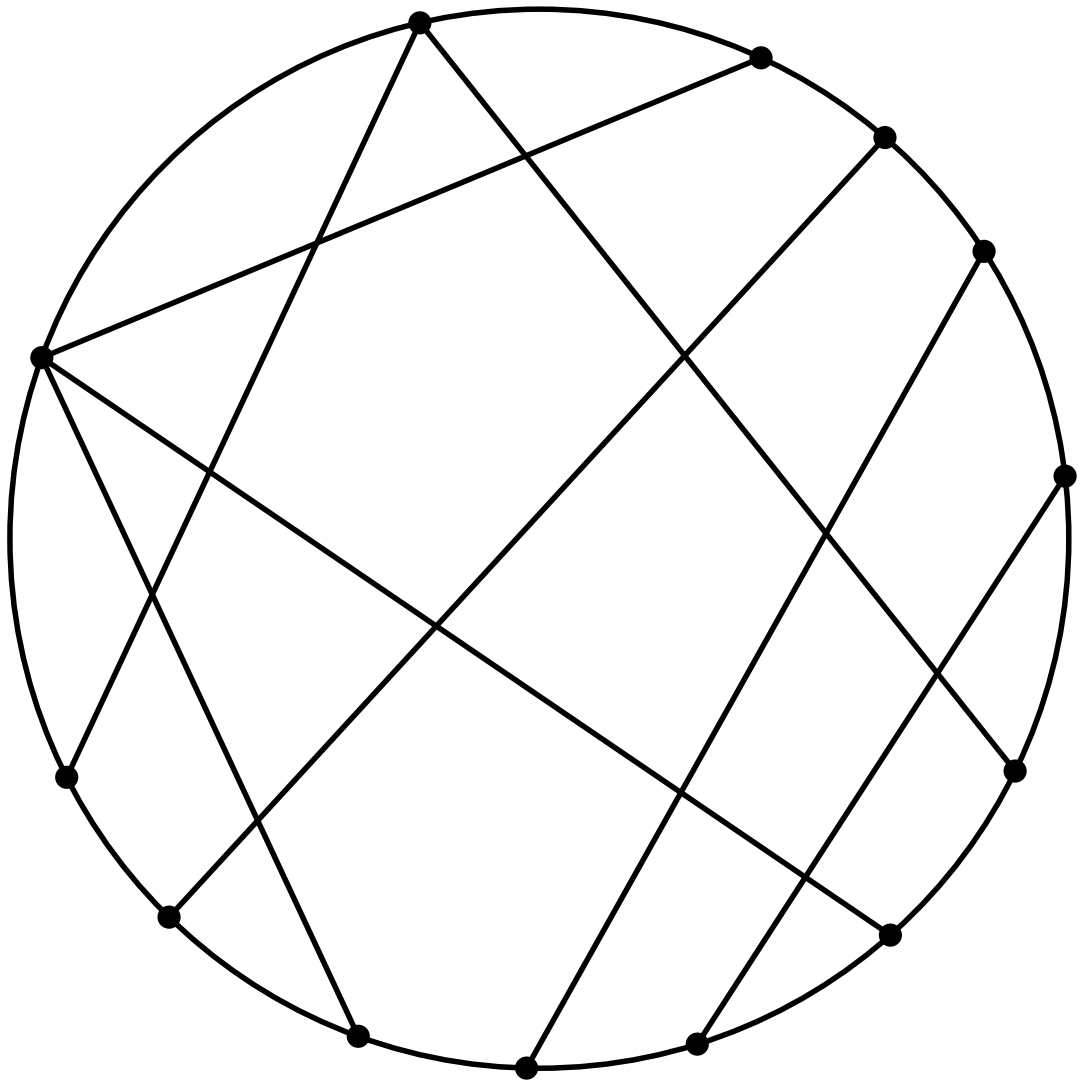
Let G be an outer min-k-planar graph. Then $\mathrm{tw}(G) \leqslant 3 \cdot \lfloor k/2 \rfloor + 4$.

Theorem

Let G be an outer min-k-planar graph. Then $\mathrm{sn}(G) \leqslant 2 \cdot \lfloor k/2 \rfloor + 4$.

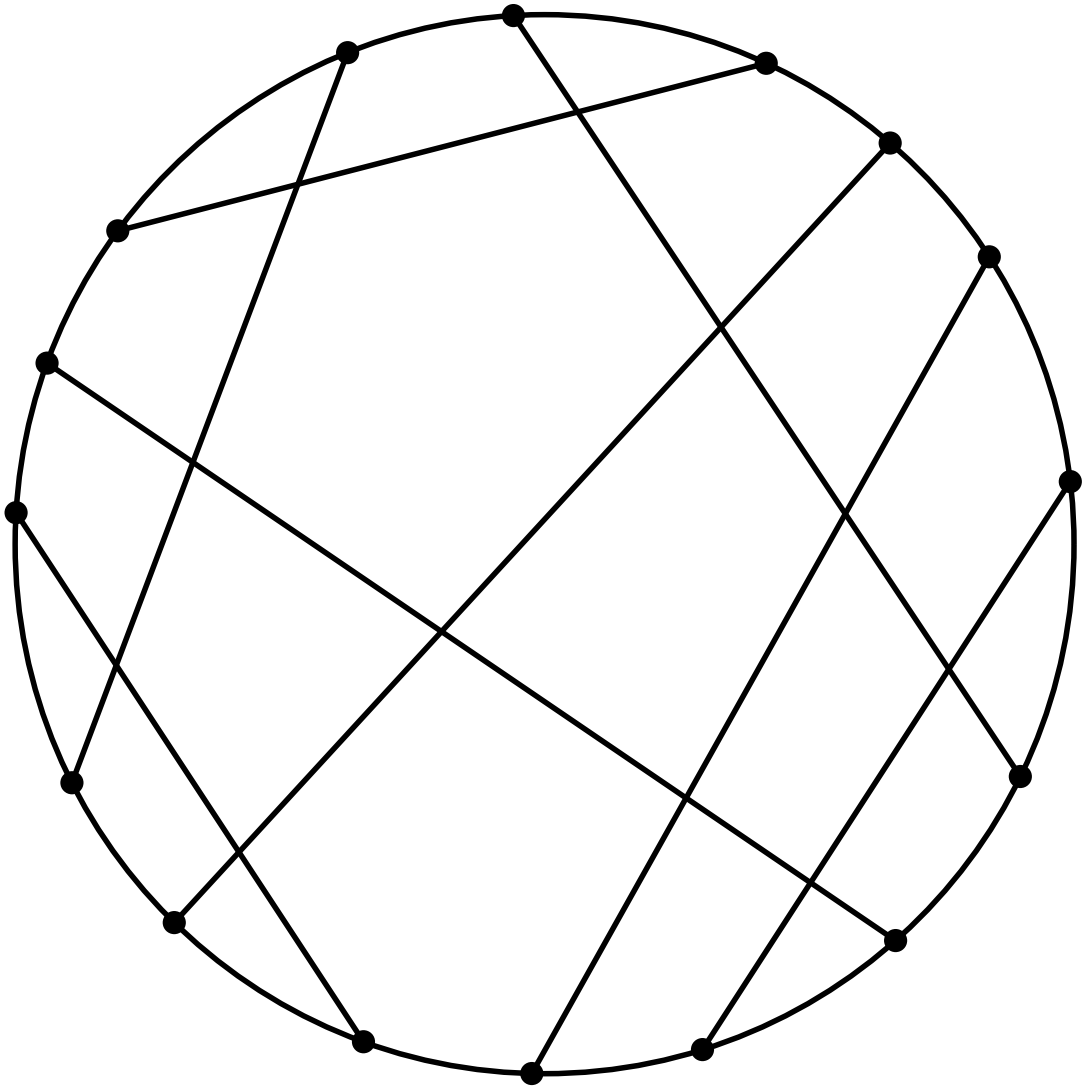
Treewidth upper bound sketch

We consider a graph that is maximal



Treewidth upper bound sketch

We consider a graph that is maximal and expanded.



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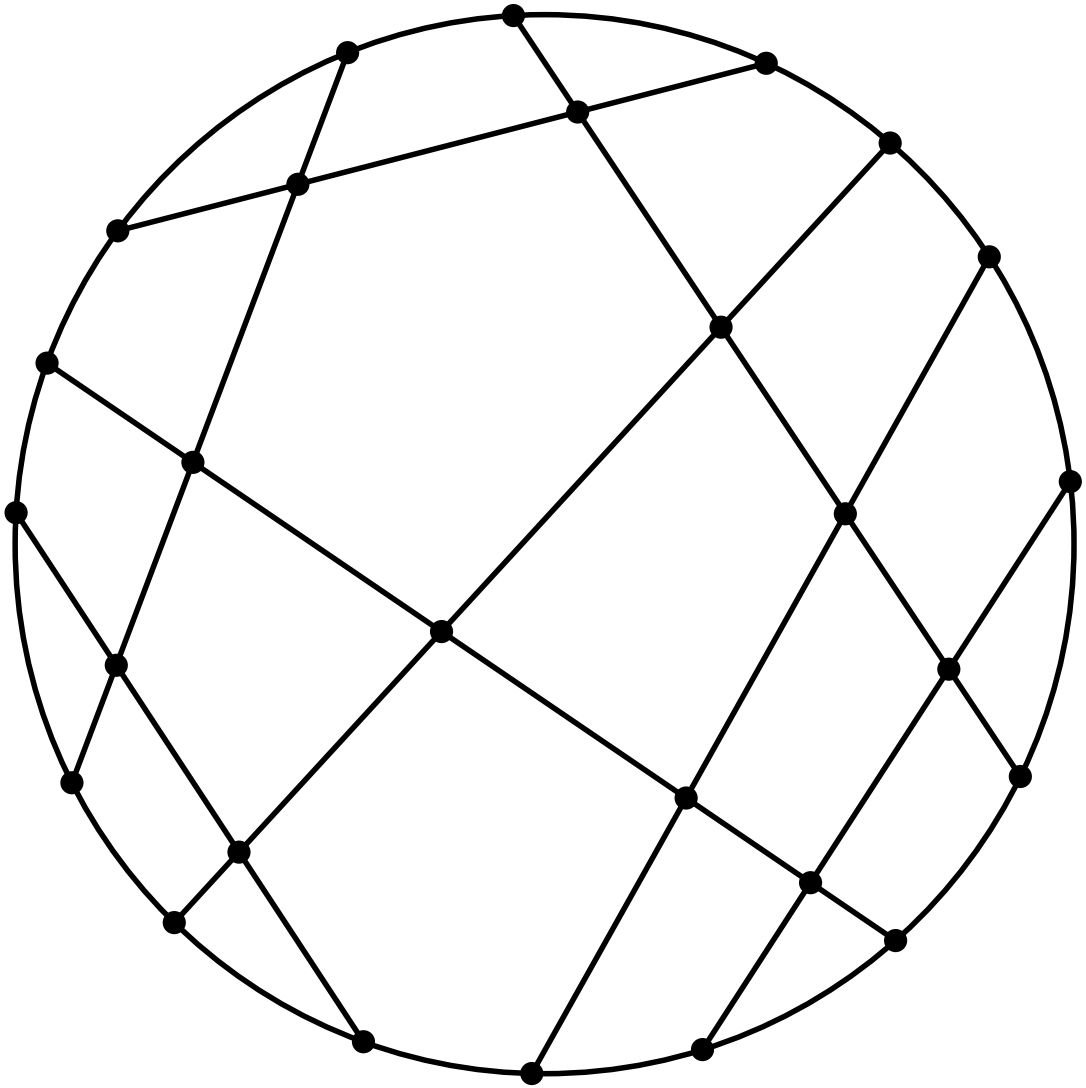

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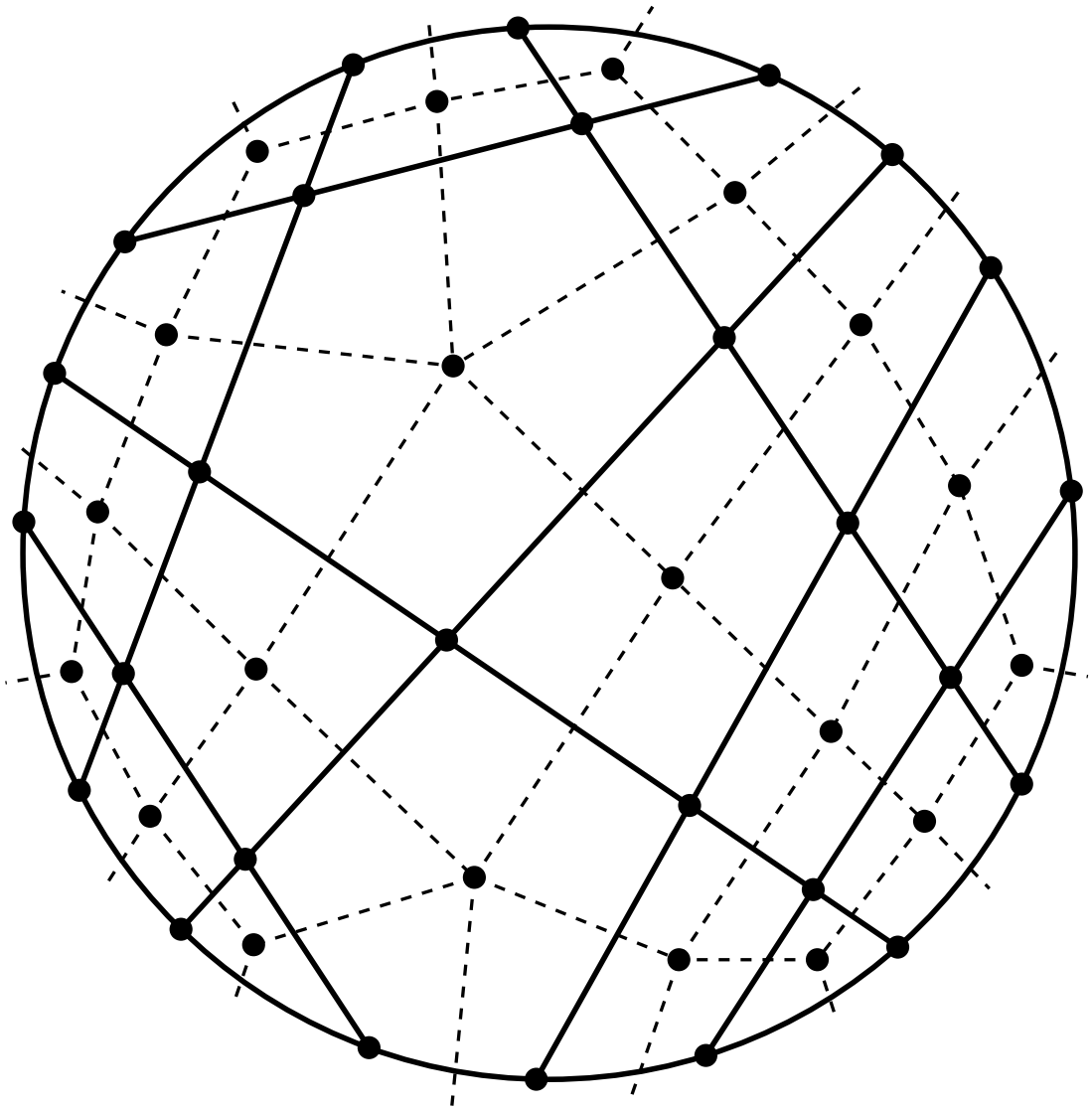
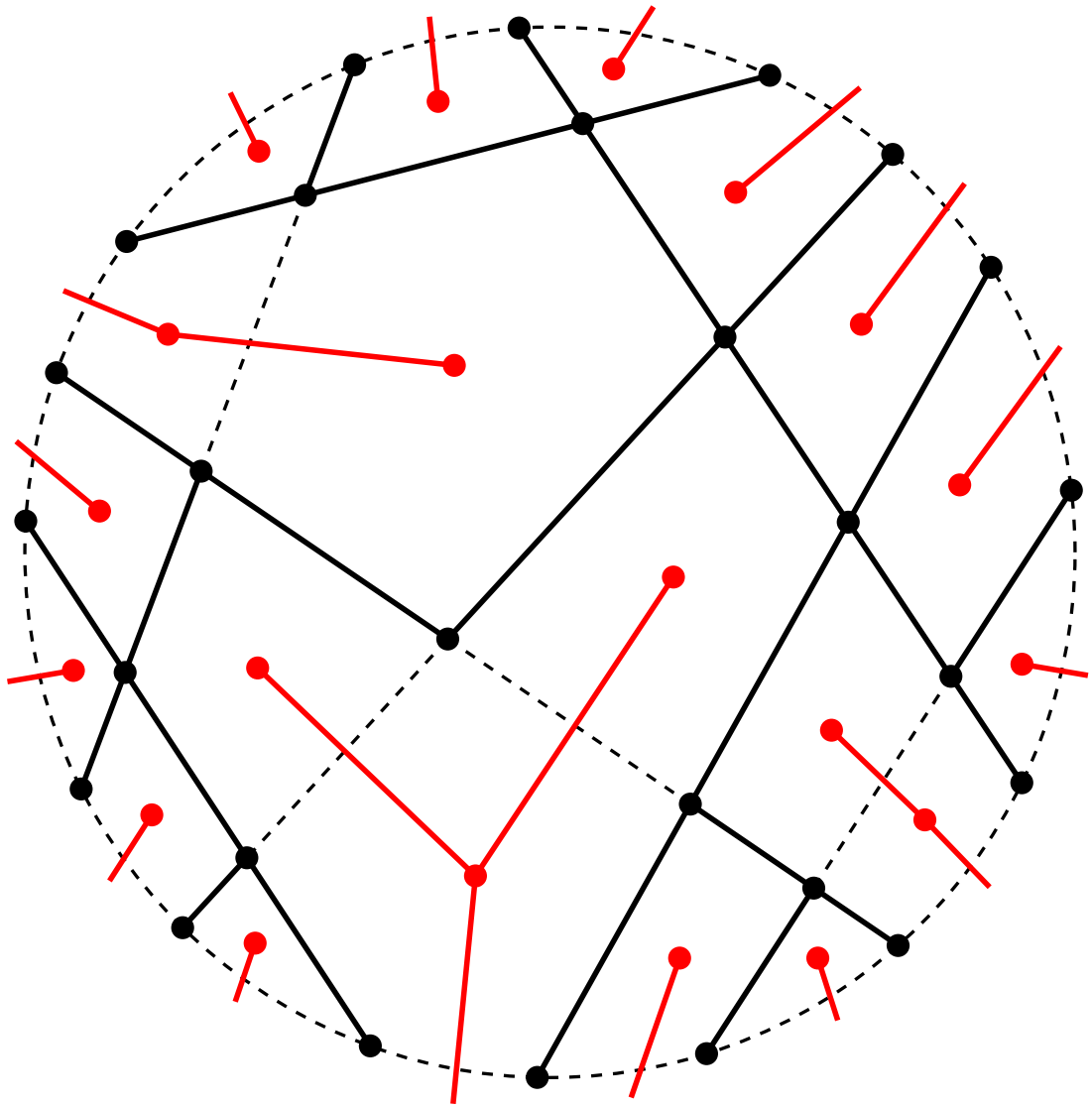


Treewidth upper bound sketch

We consider a graph that is maximal and expanded.

Distance of any red vertex to the outer face is $\leqslant k/2 + 1$.



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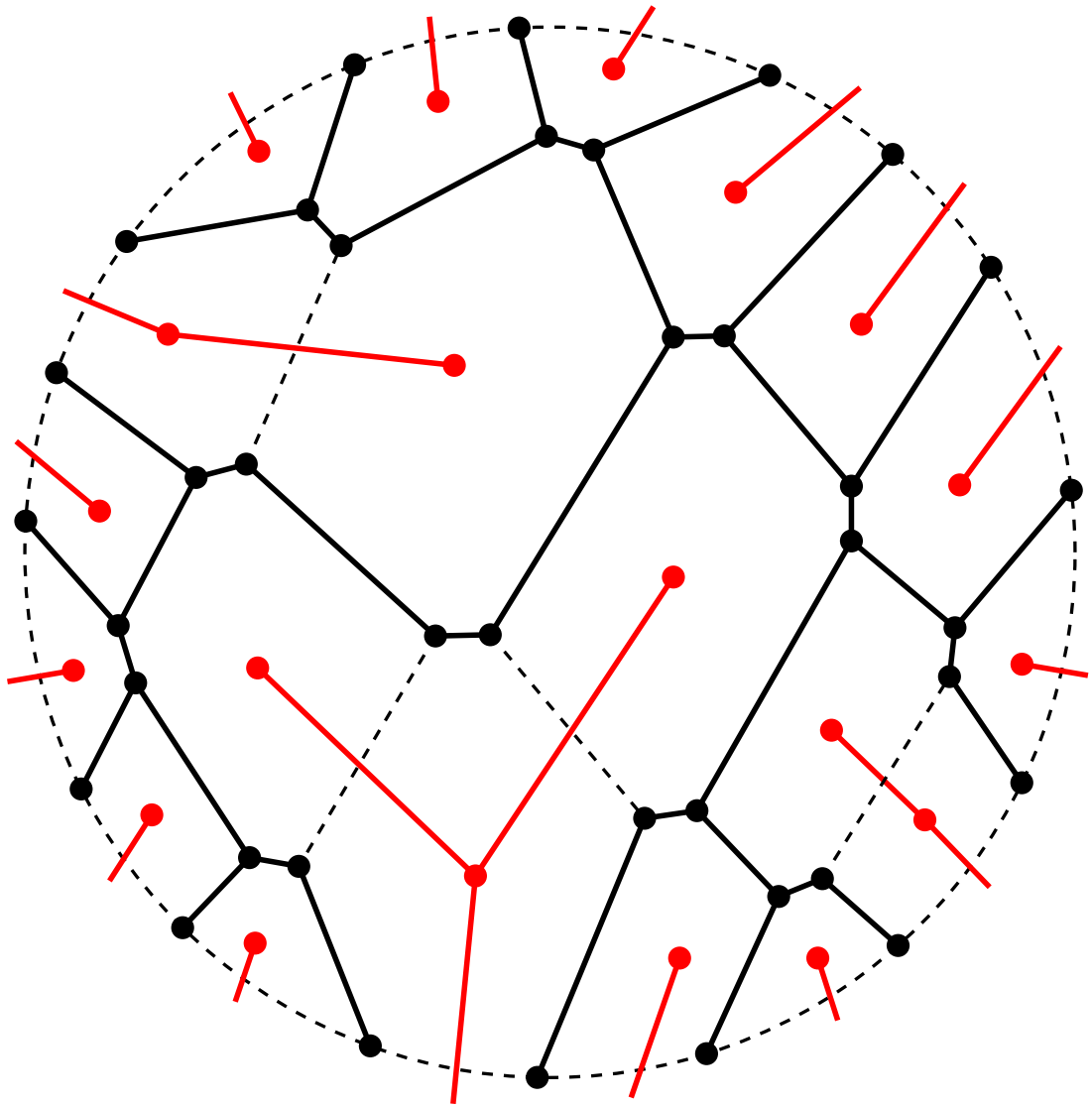

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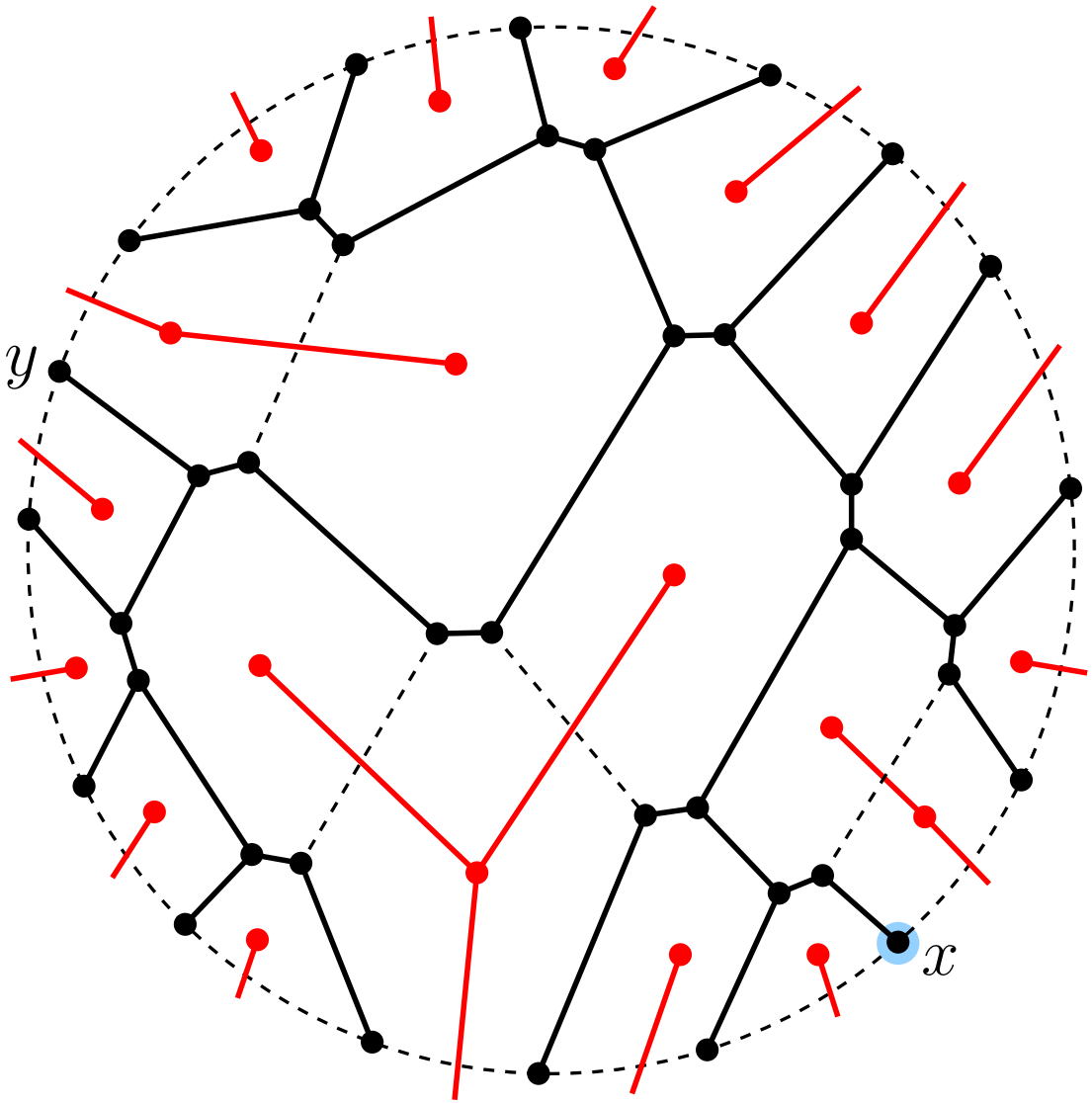
We place vertex x into bags corresponding to the blue path.



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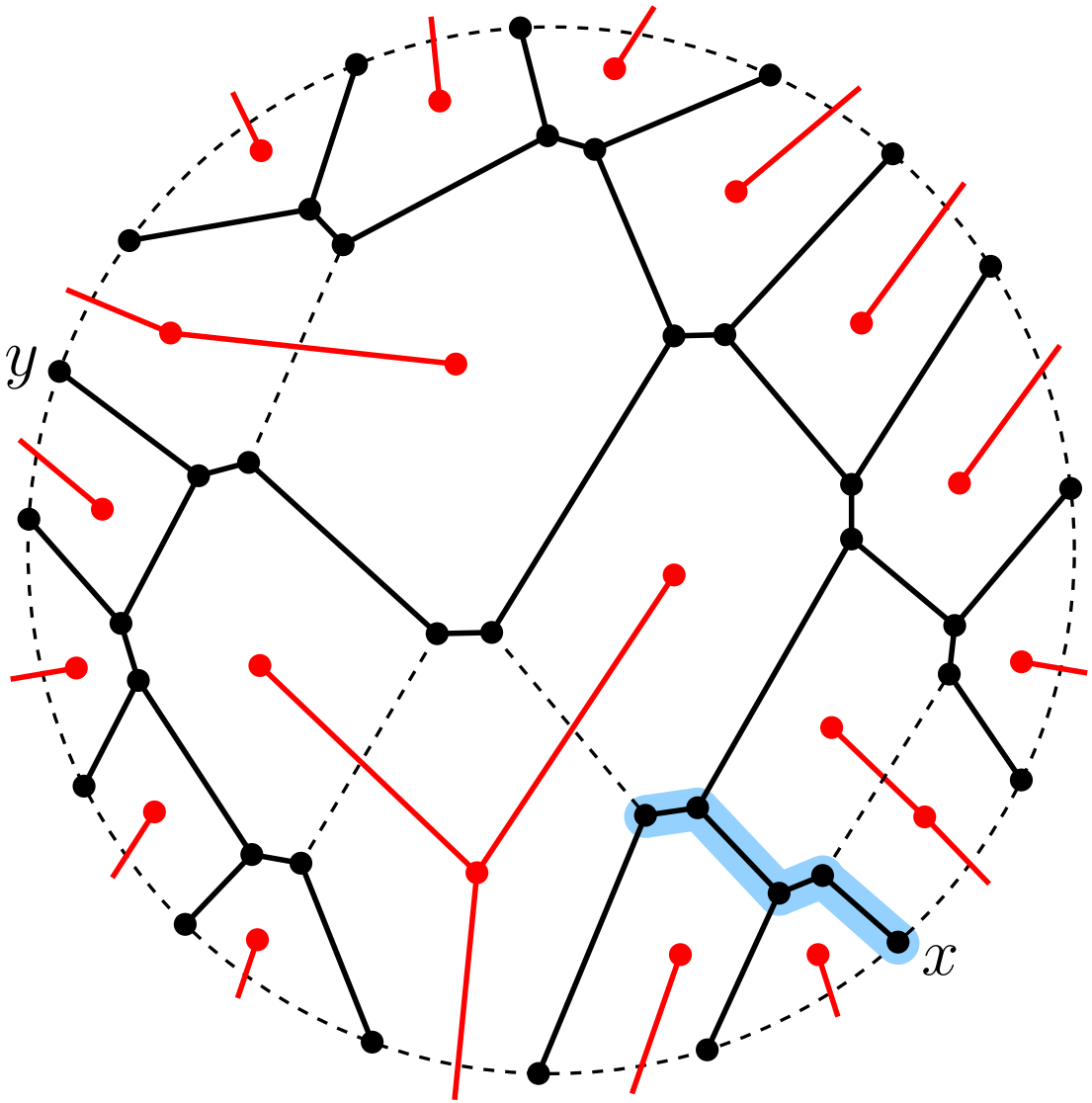

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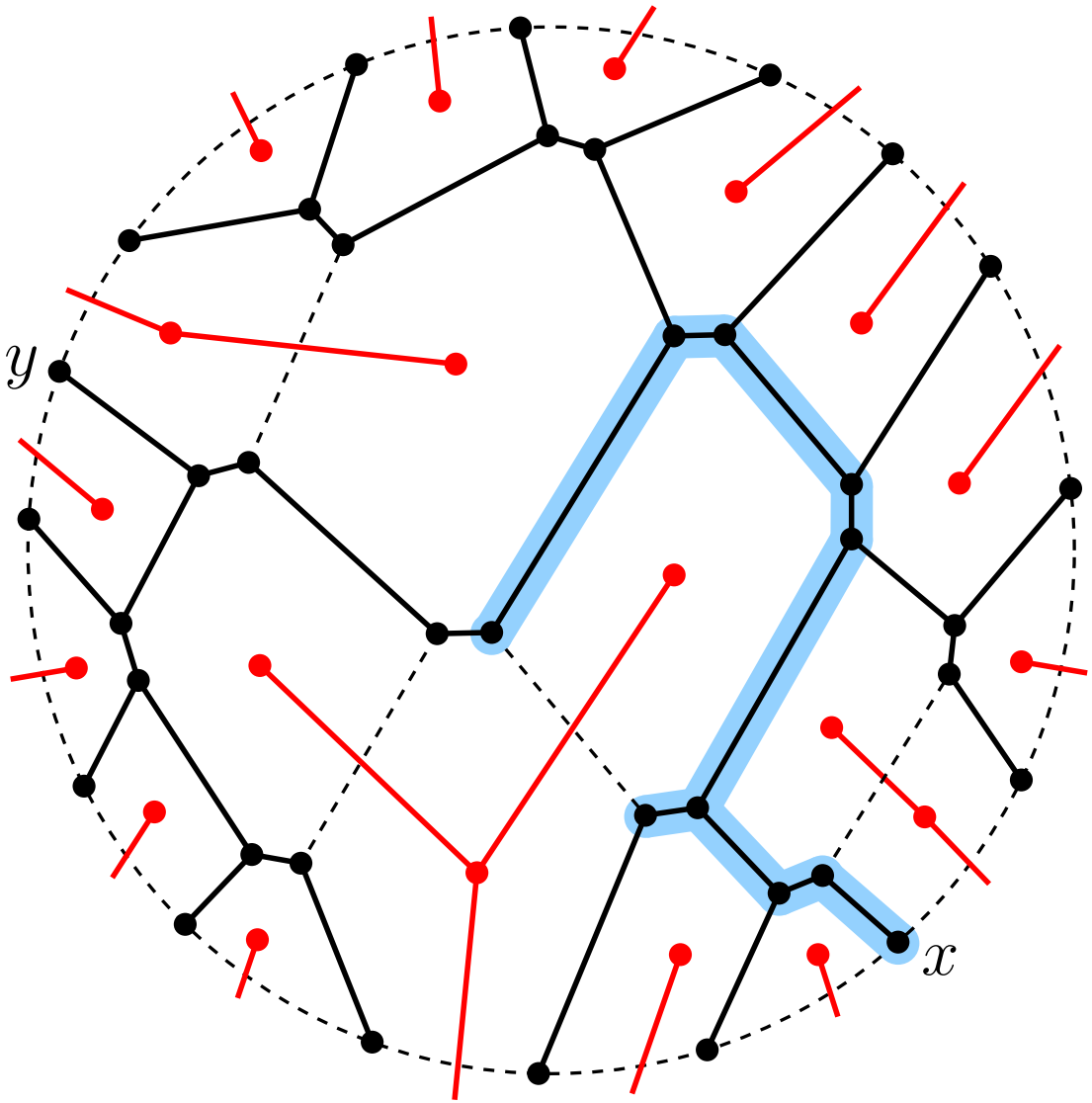

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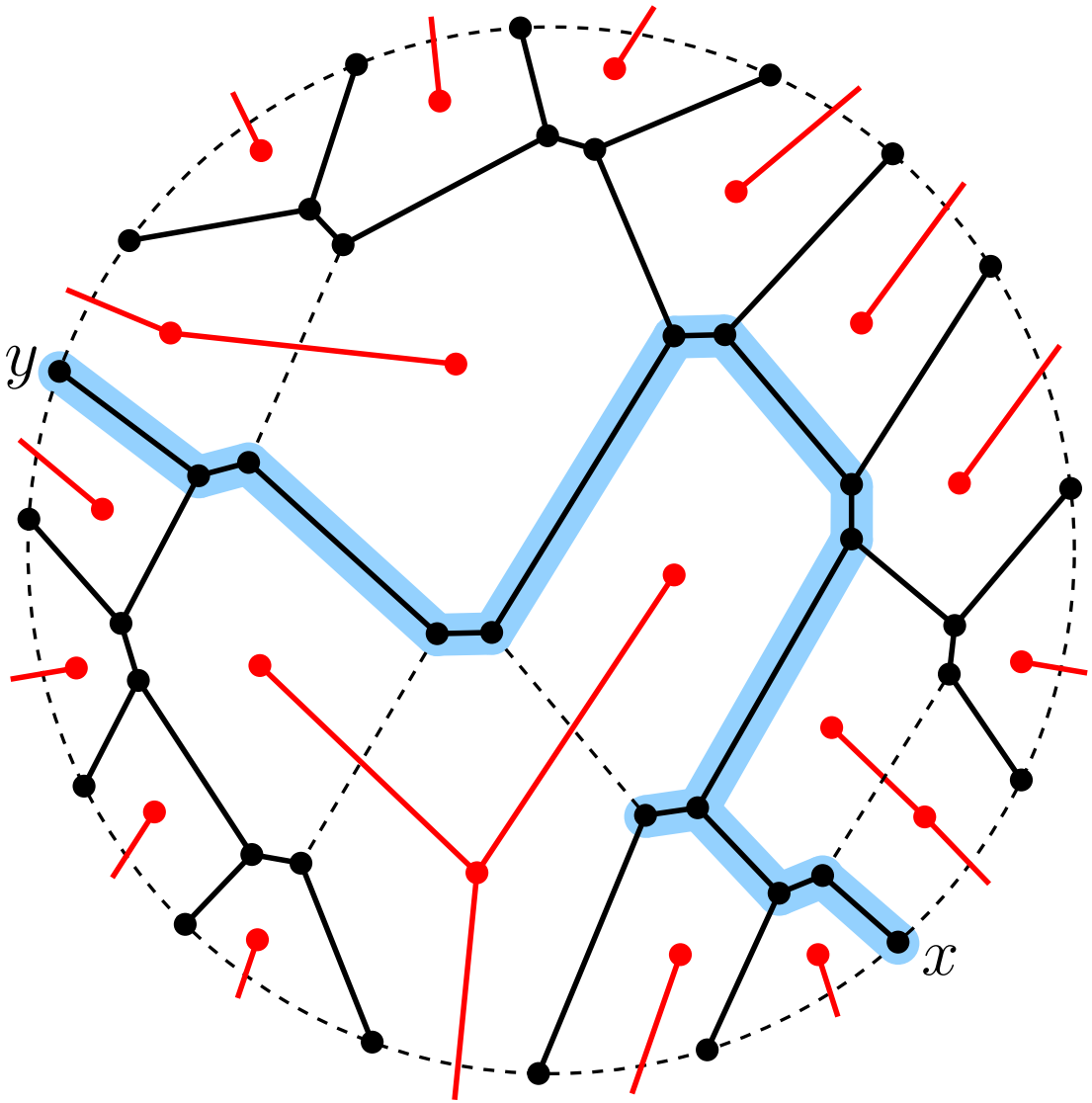

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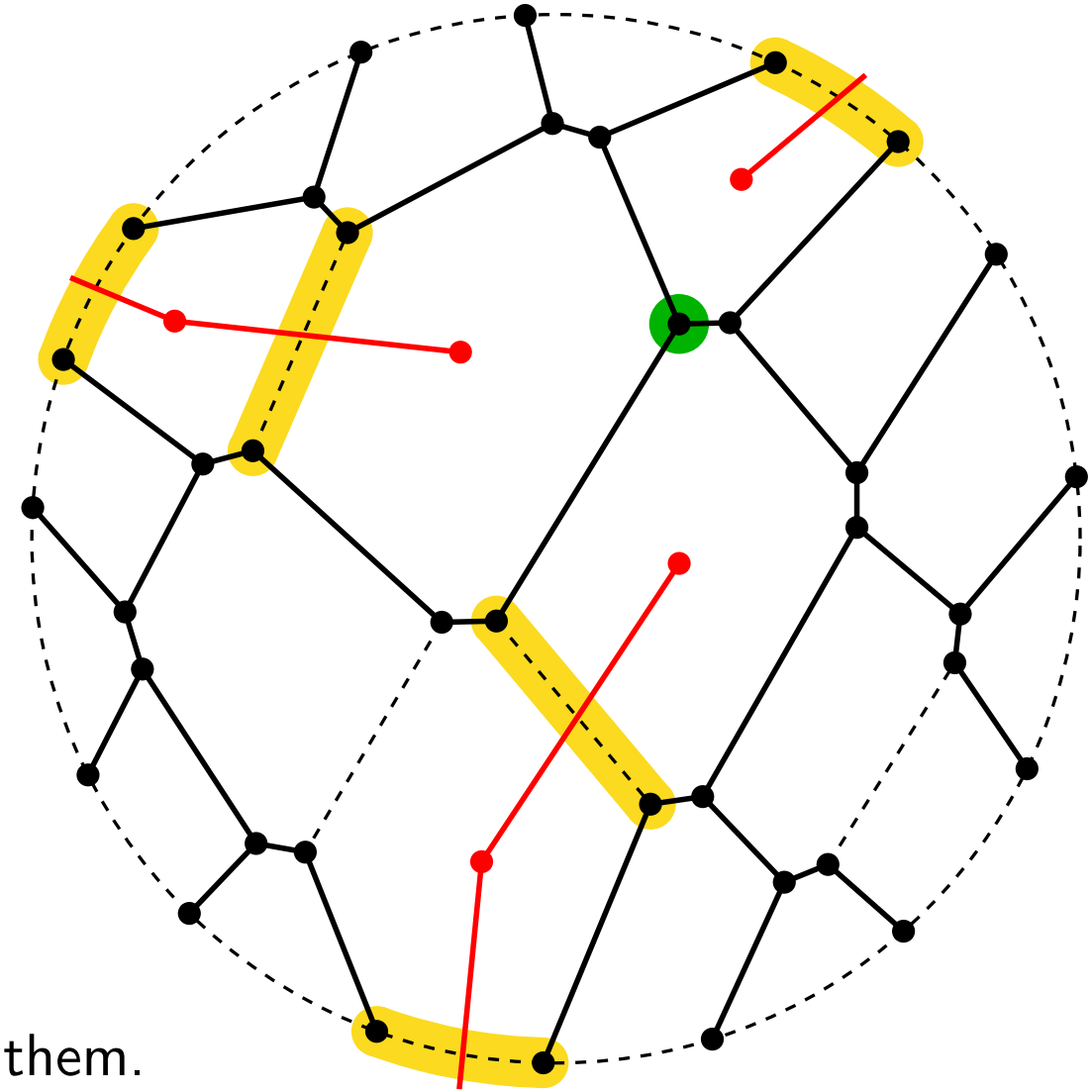
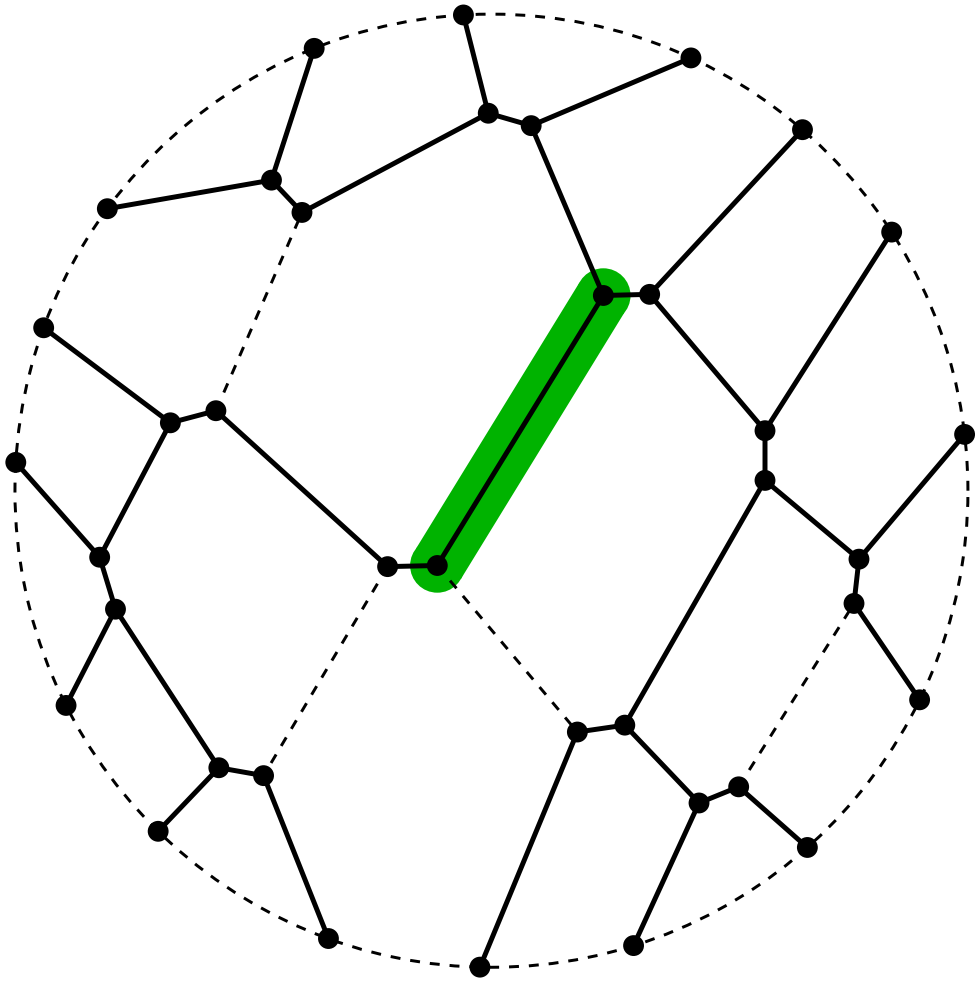
Extra vertices placed into the green bag correspond to yellow edges.

There are at most $1.5k + 3$ of them.



Separation number upper bound sketch

Structural property:
upper bound the adhesion
of the tree decomposition



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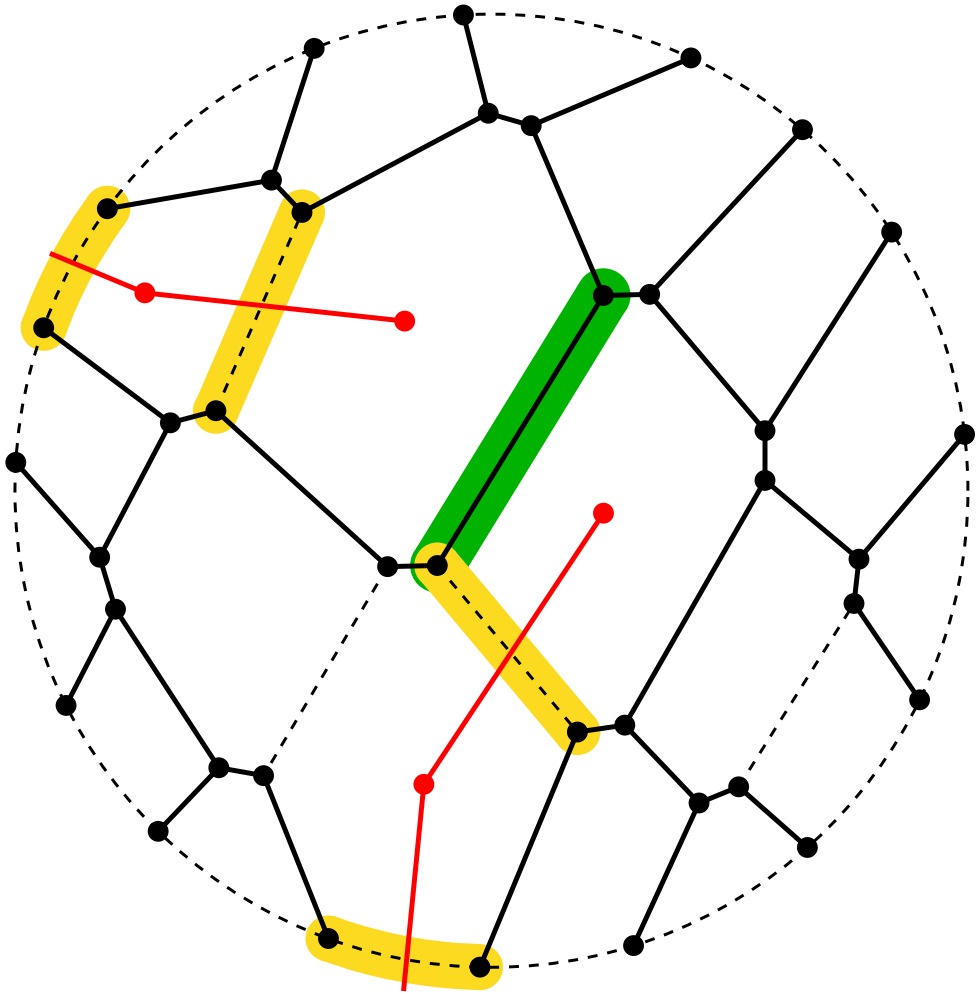

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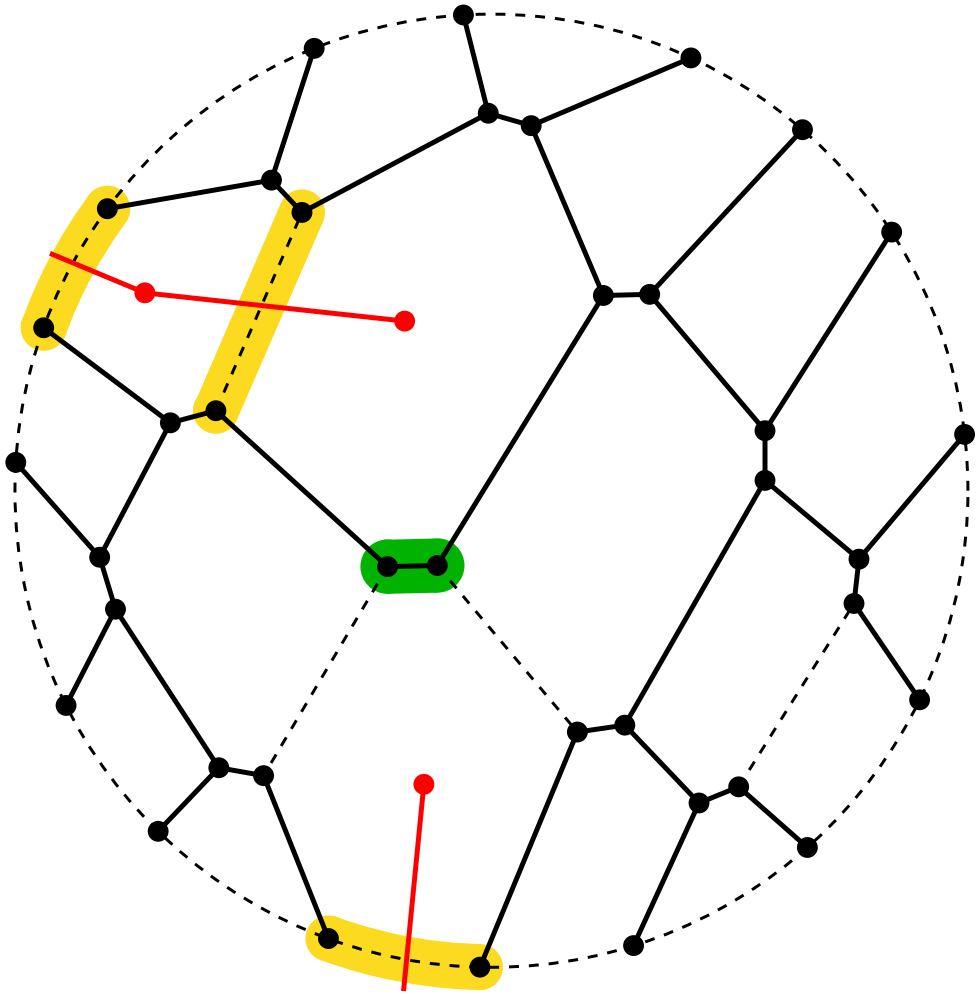

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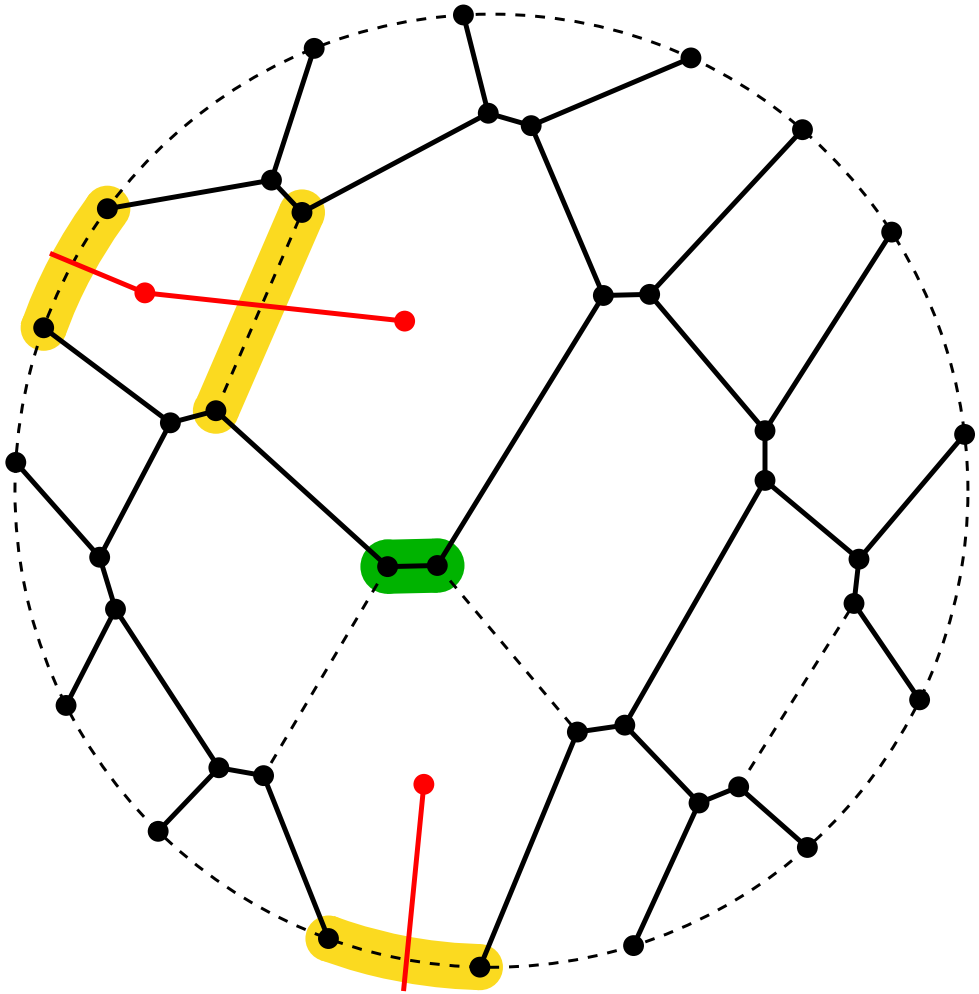

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Revert expansion:
obtain a tree decomposition
for every outer min-k-planar graph.

Structural property still holds.