Internally-Convex Drawings of Outerplanar Graphs in Small Area

Michael A. Bekos, Giordano Da Lozzo, Fabrizio Frati,
Giuseppe Liotta, **Antonios Symvonis**

Graph Drawing 2025

Introduction

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What is the paper about?

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- Planar straight-line grid drawings

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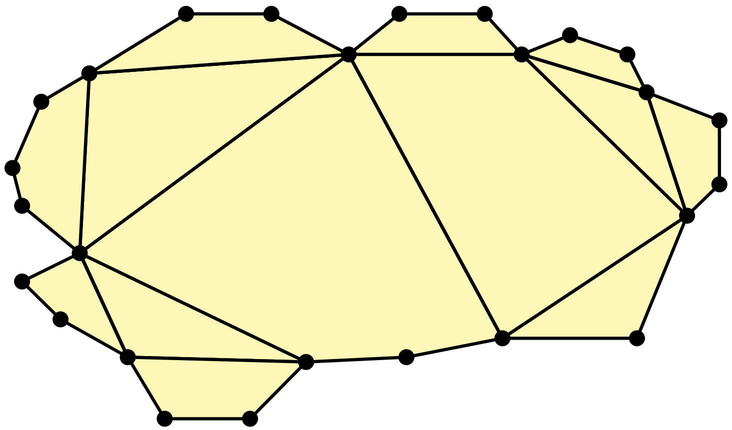
- Outerplanar graphs
  We want: Internally convex drawings



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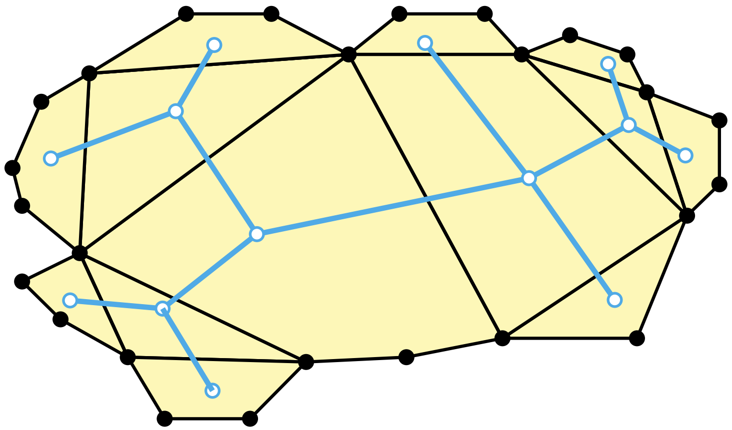

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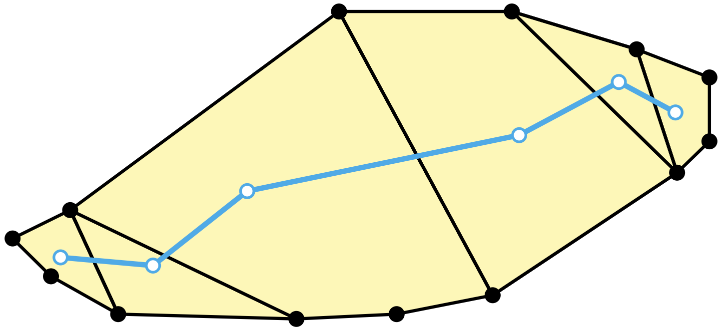

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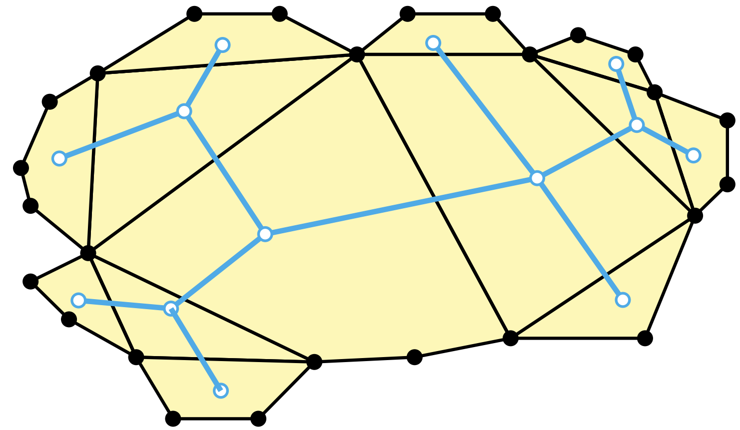

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Question:

How big is the drawing?

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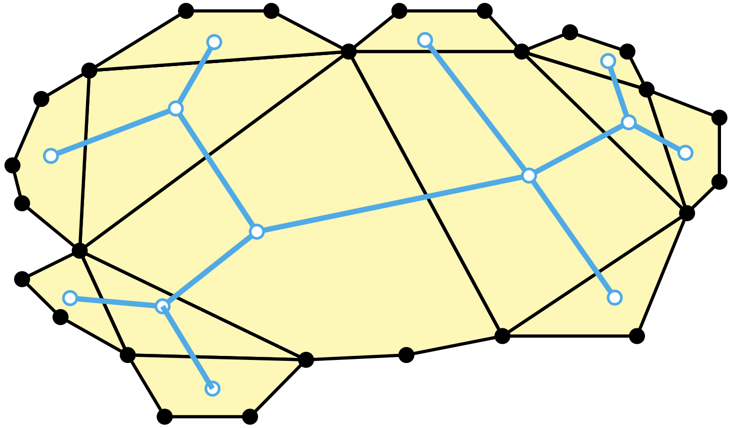

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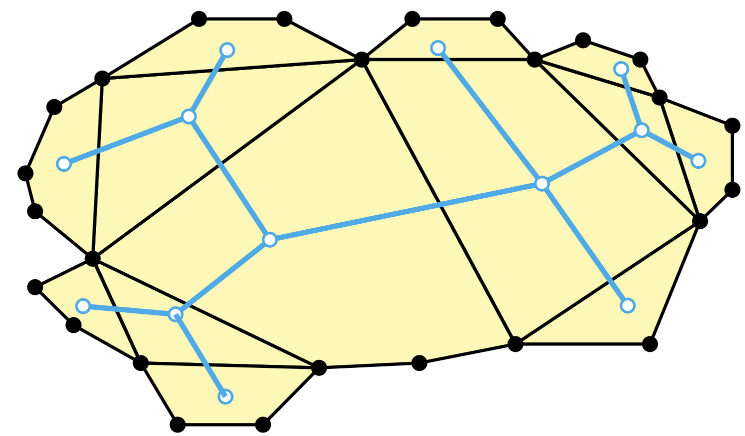

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The State of the Art:

Chrobak and Kant (1997)
Felsner (2001)

Every 3-connected plane graph admits a convex planar straight-line grid drawing of $O(n^2)$ area

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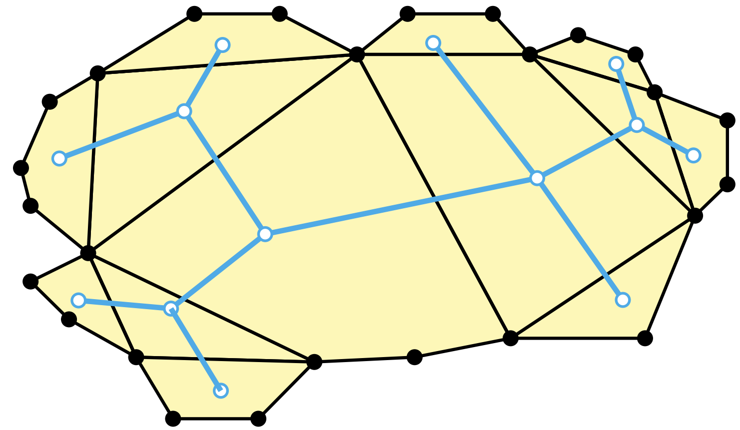

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This paper:

BDFLS (2025)

Every n-vertex outerplane graph admits an embedding-preserving internally-convex grid drawing in $O(n^{1.5})$ area

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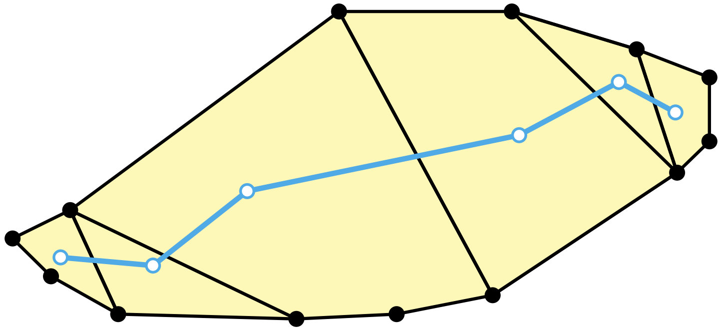

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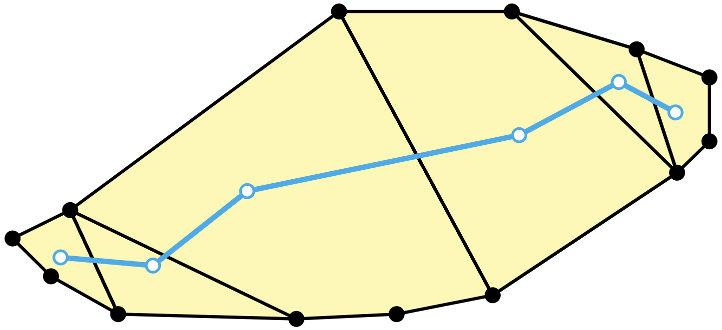

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The State of the Art:

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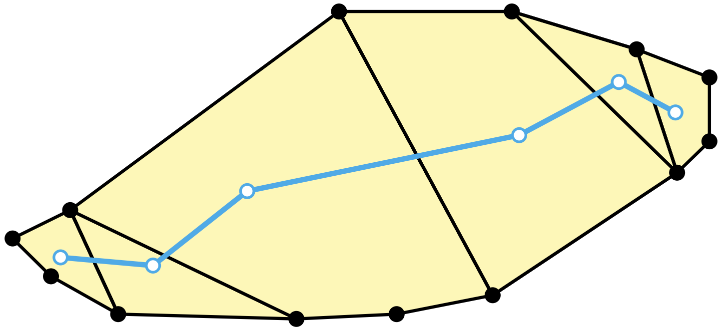

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Every n-vertex outerpath whose internal faces have size at most k admits an internally-strictly-convex grid drawing in $O(nk^2)$ area.

Preliminaries

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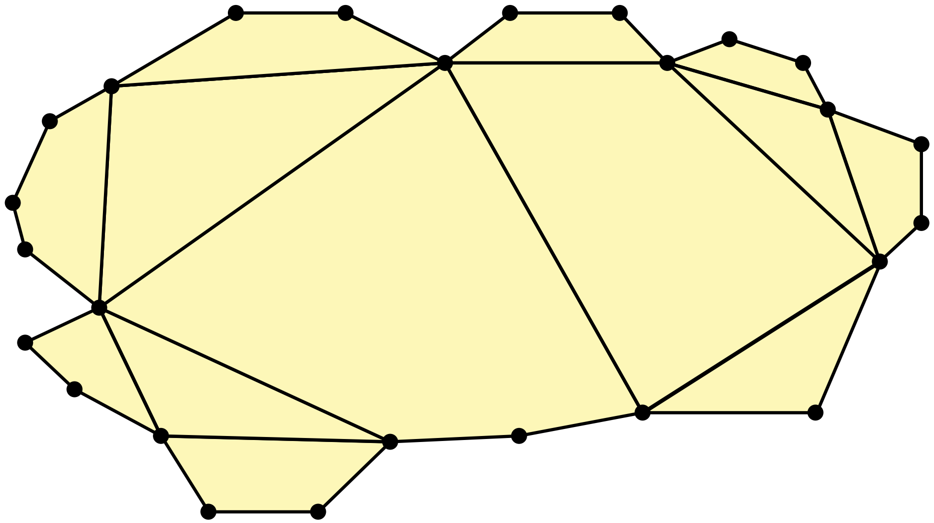

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- Outerplanar graph
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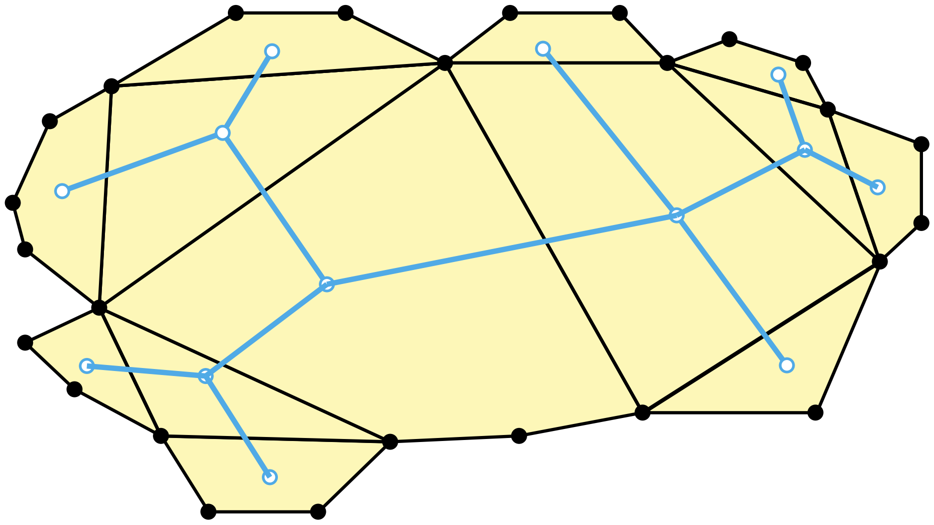

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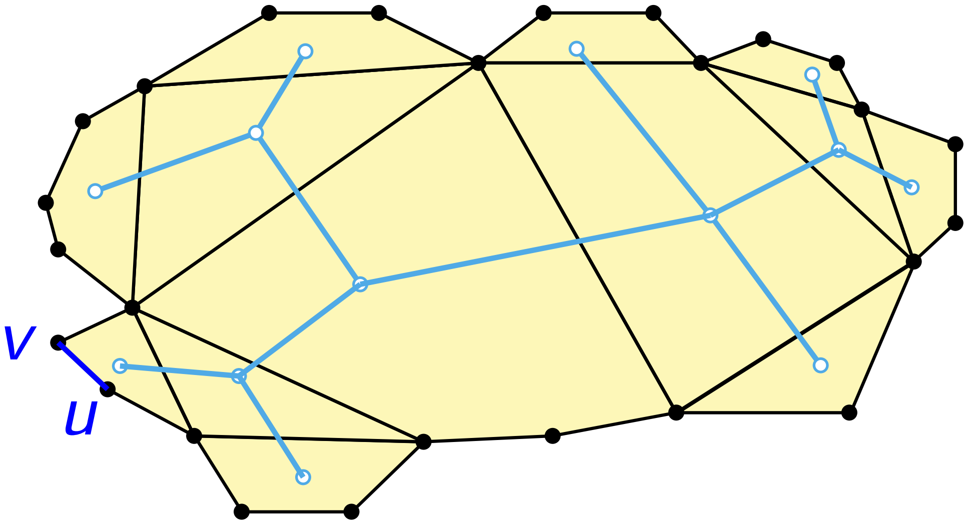

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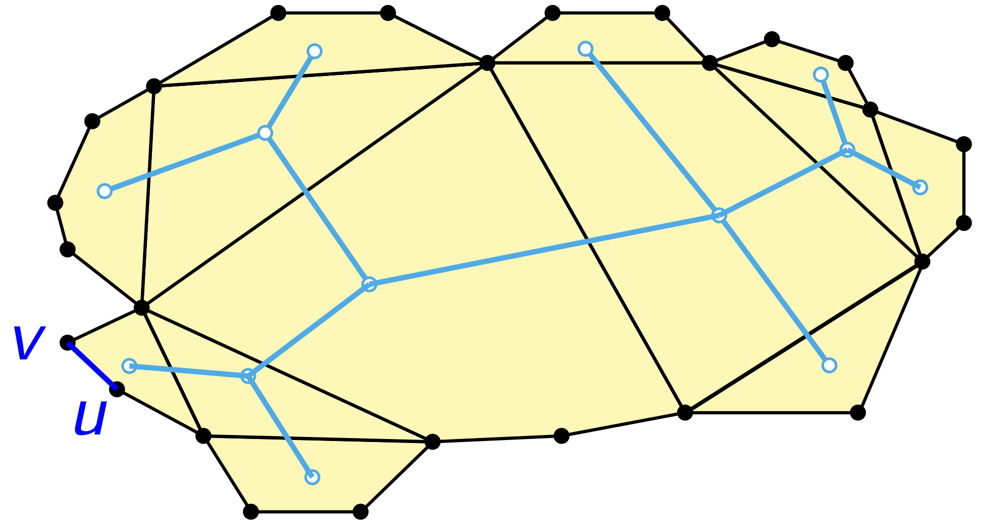

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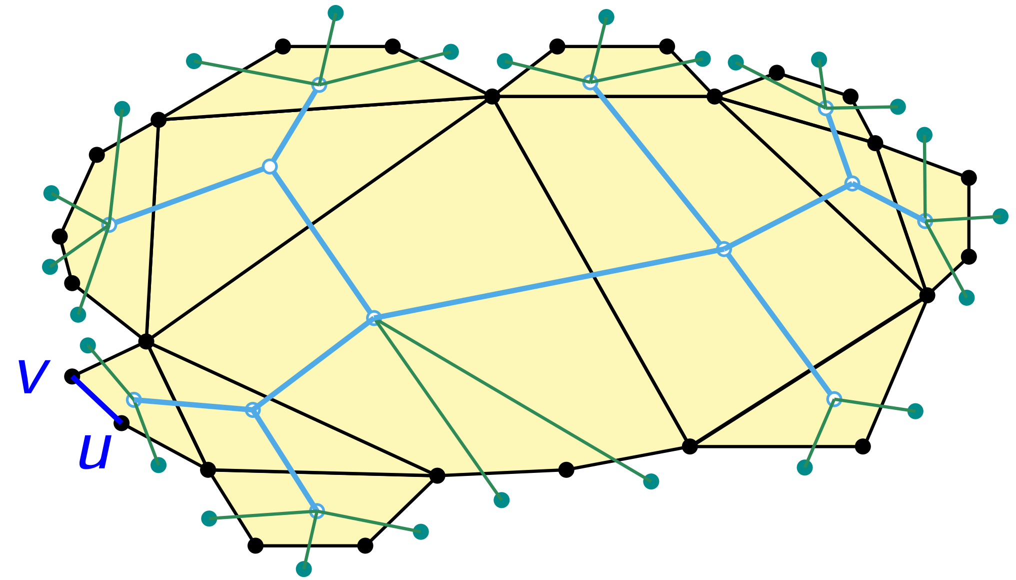

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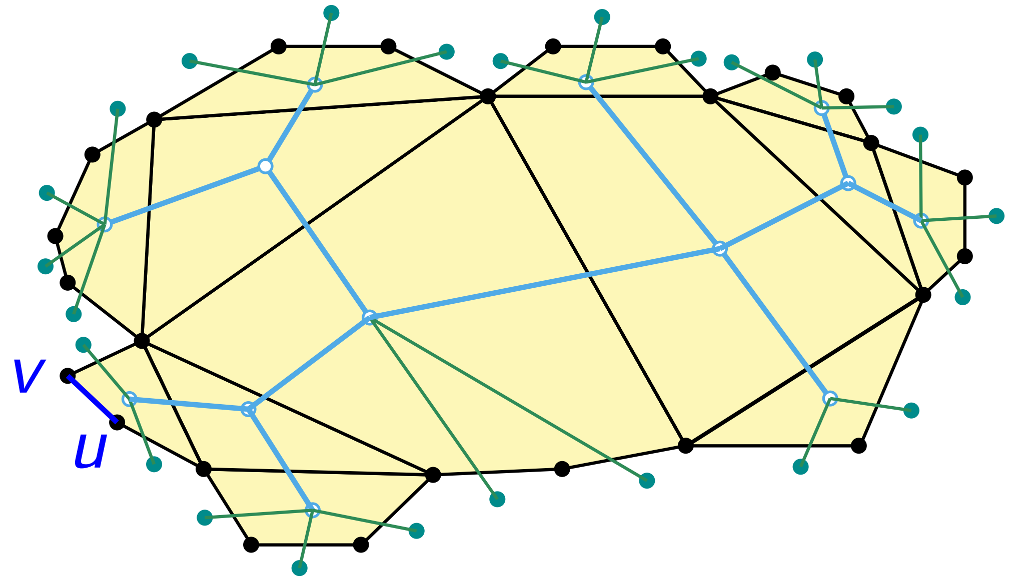
  $|T| = d \leq 2n - 3$



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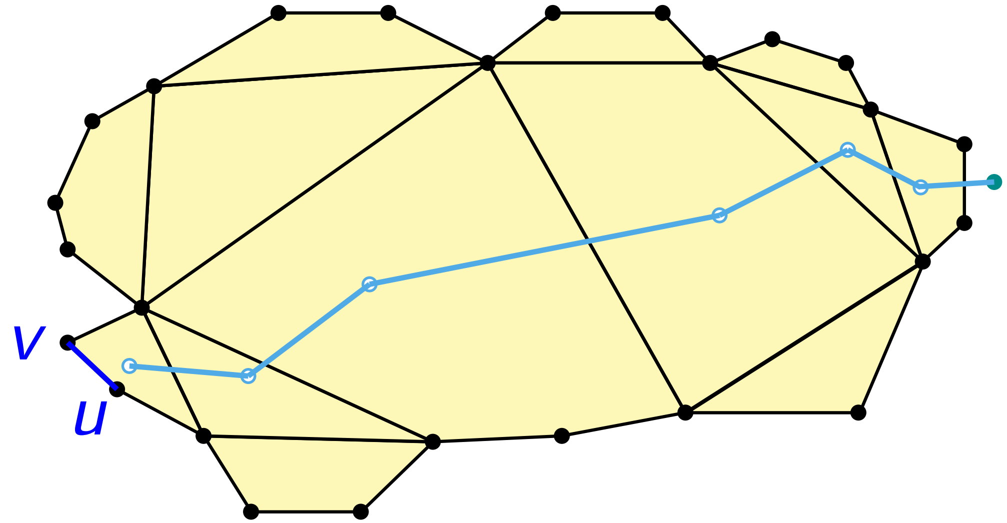

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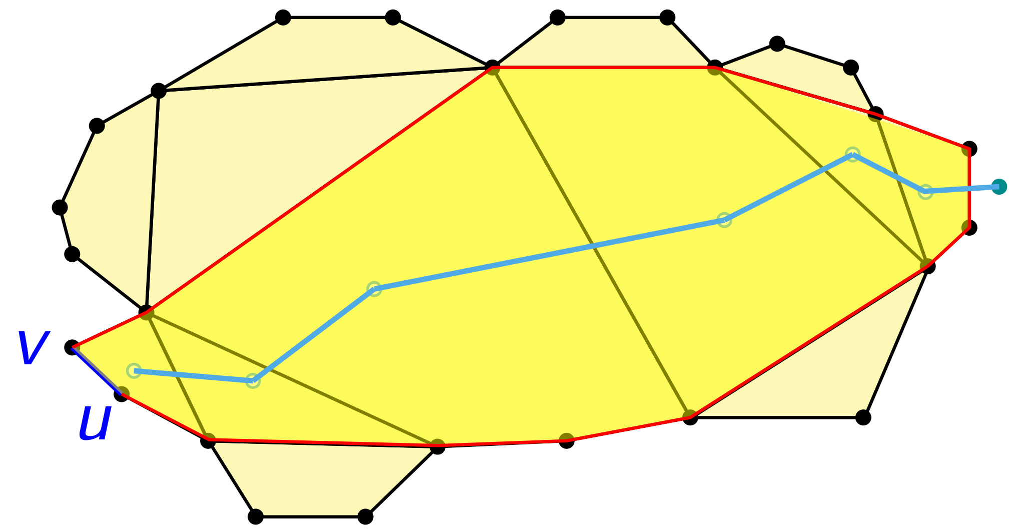

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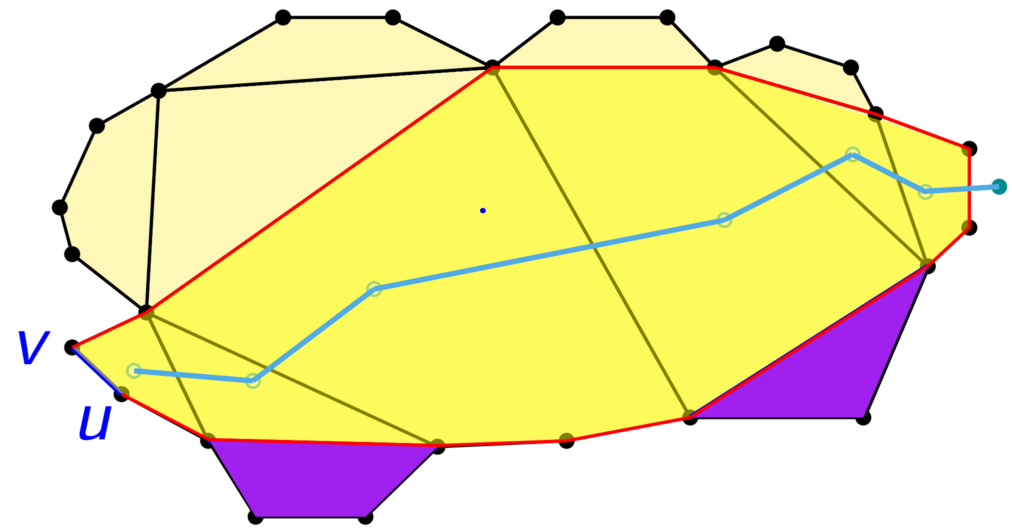

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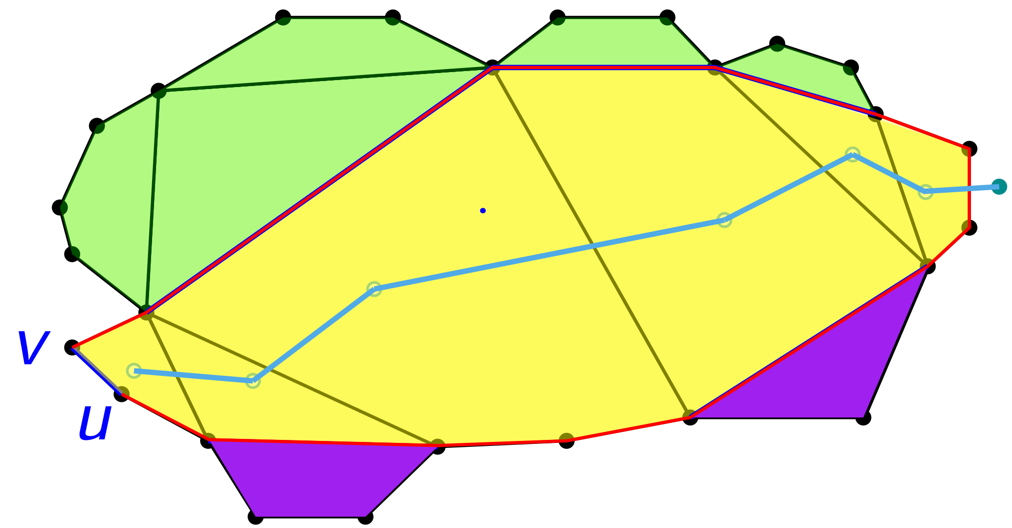

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Biedl, Liotta, Lynch, Montecchiani (BLLM-2024)

Let $p = 0.48$. Given any rooted tree with n vertices, there exists a root-to-leaf path π such that for any left subtree α and for any right subtree β of π, $|\alpha|^p + |\beta|^p \leq (1 - \delta)n^p$, for some constant $0 < \delta < 1$.

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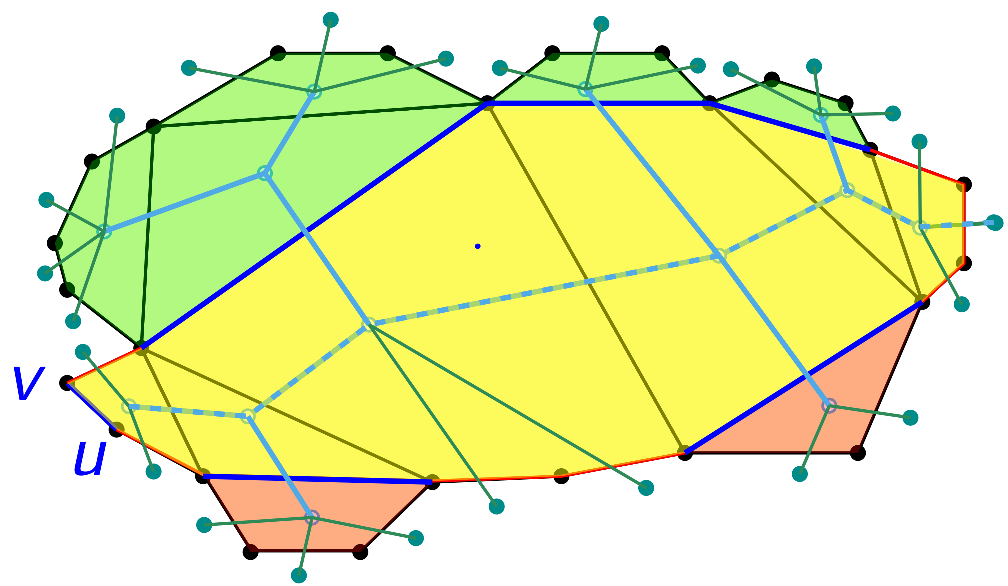

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Theorem

Let $f(d)$ be the recursive function defined on $\mathbb{N}_0$ as follows:

- $f(0) = 0$ and $f(1) = 1$.
- $f(d) = \max\limits_{d_a, d_b \in \mathbb{N}_0 : d_a^p + d_b^p \leq (1-\delta)d^p} \{f(d_a) + f(d_b)\} + 2\sqrt{d} \qquad d \geq 2$

Then, $f(d) \leq c\sqrt{d}$.

$[c = 2/\delta,\ c > 2]$

Outerplanar graphs – Internally convex drawings

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- Some notation ...

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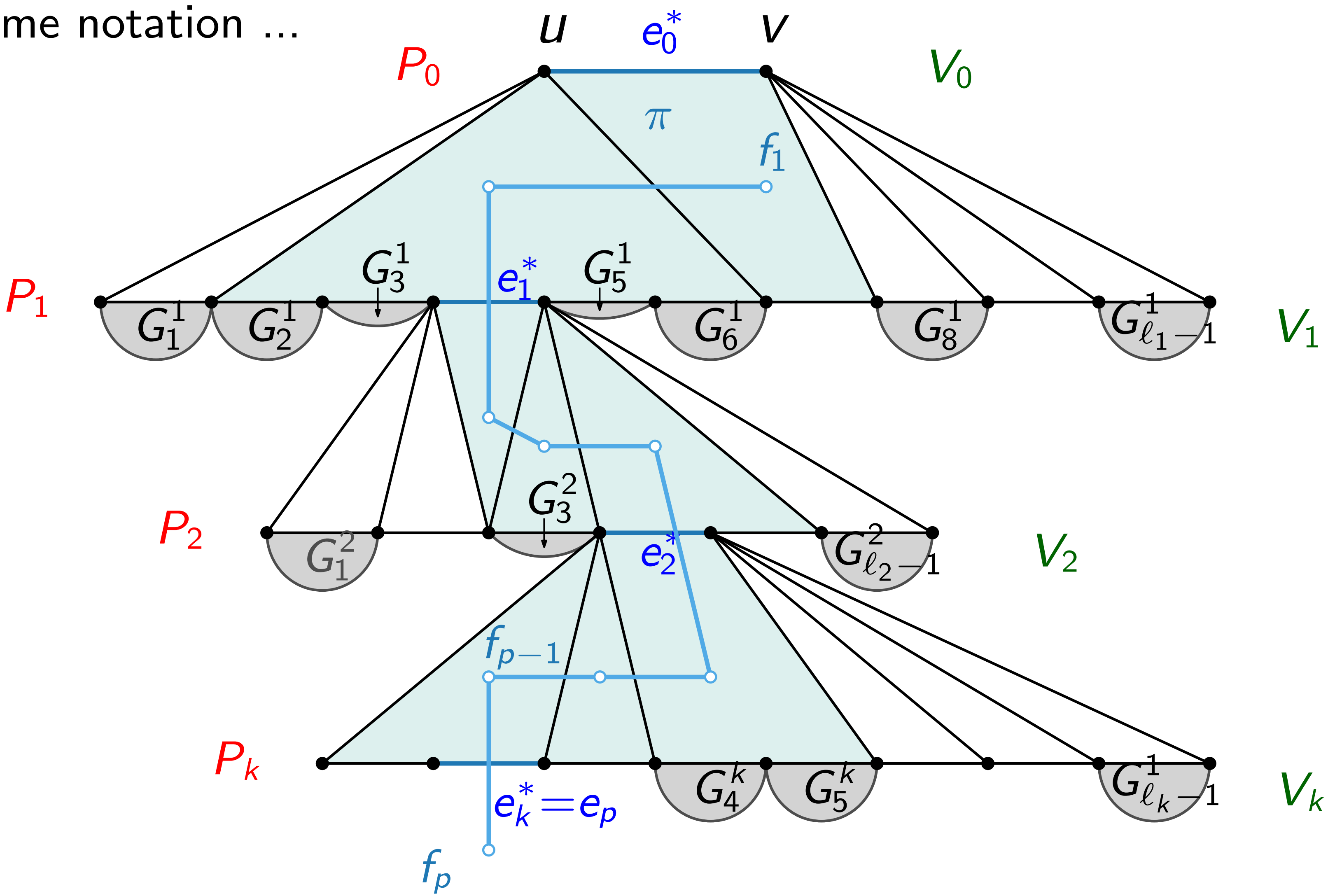
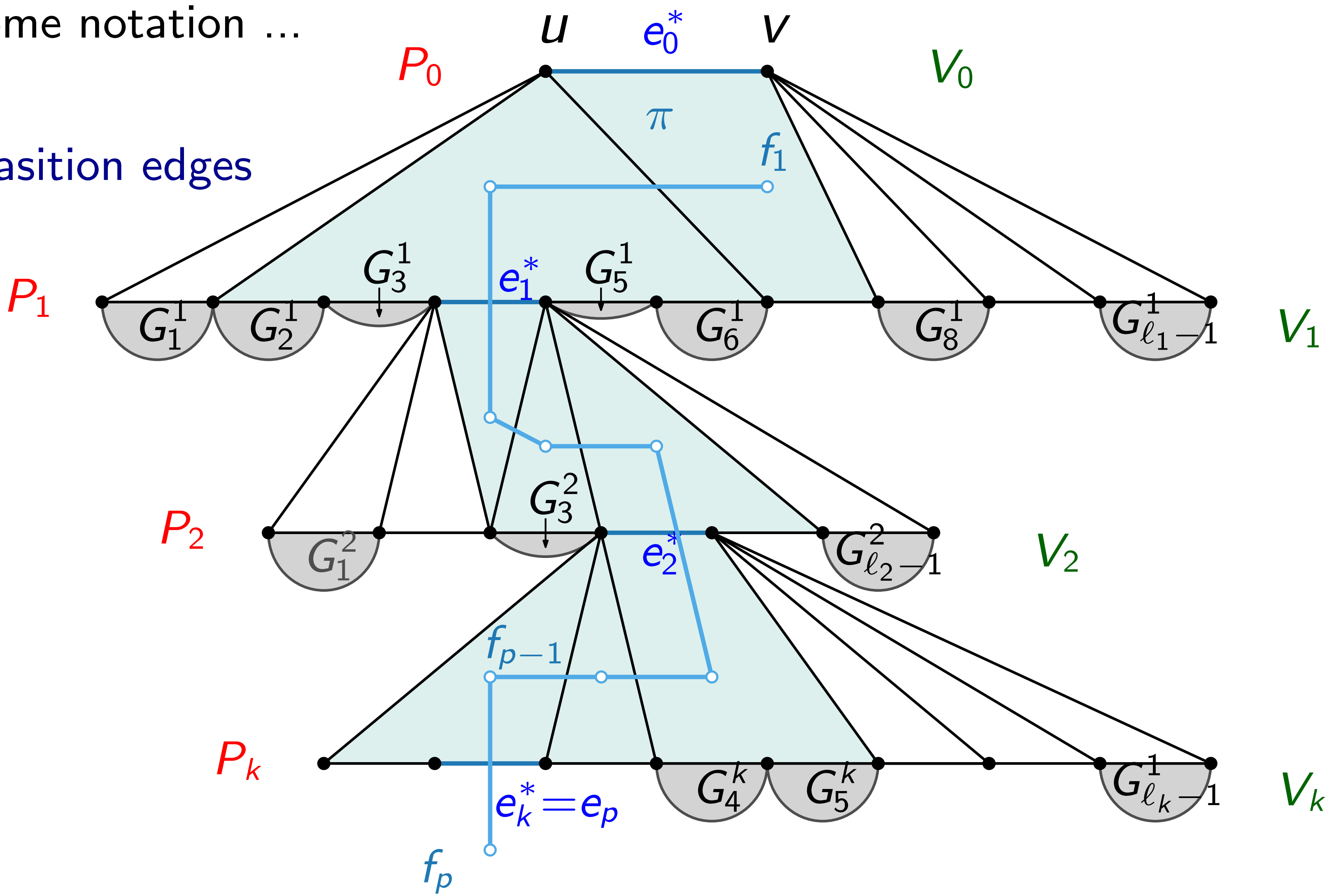


Outerplanar graphs – Internally convex drawings

- Some notation ...
- Trasition edges



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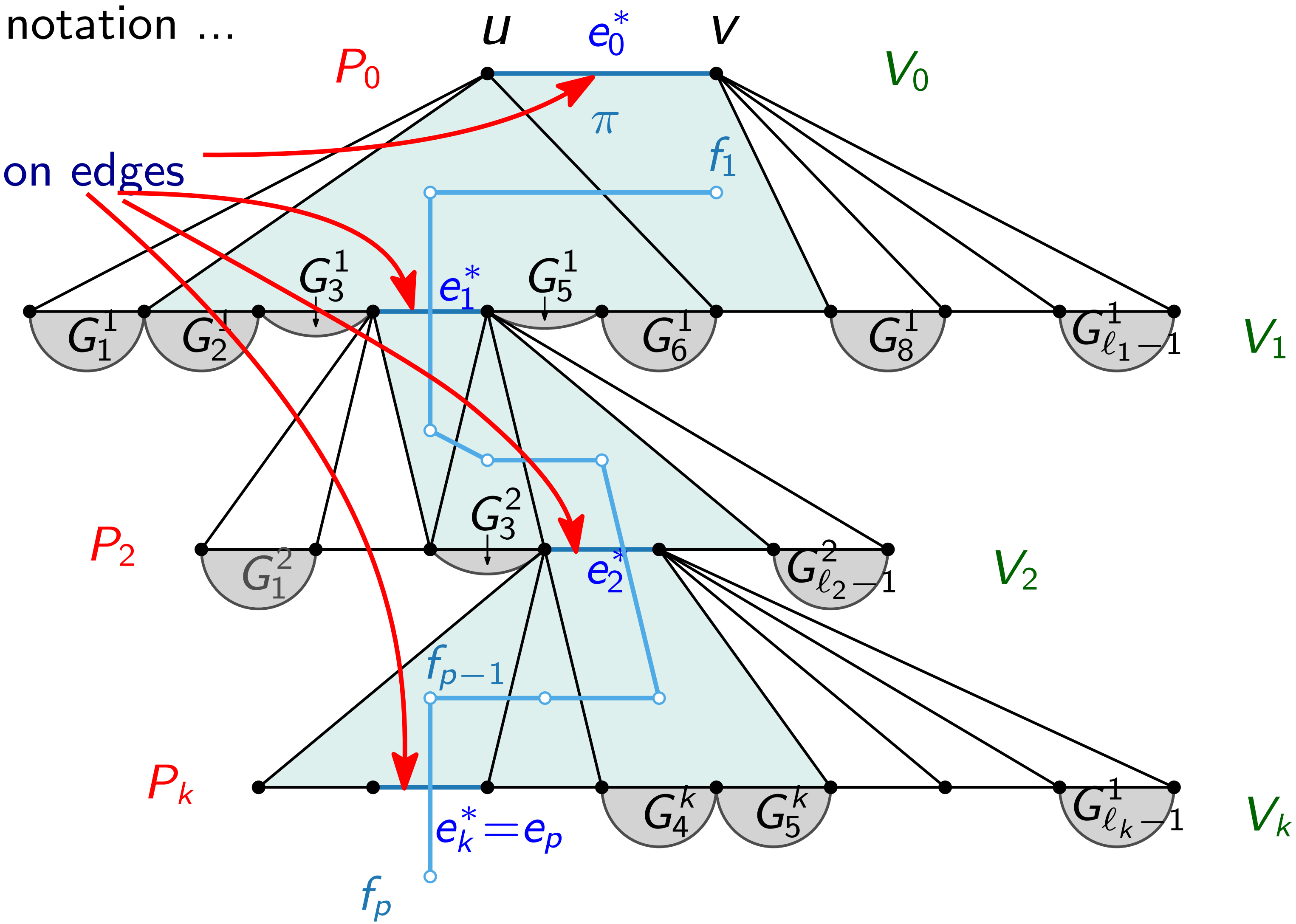

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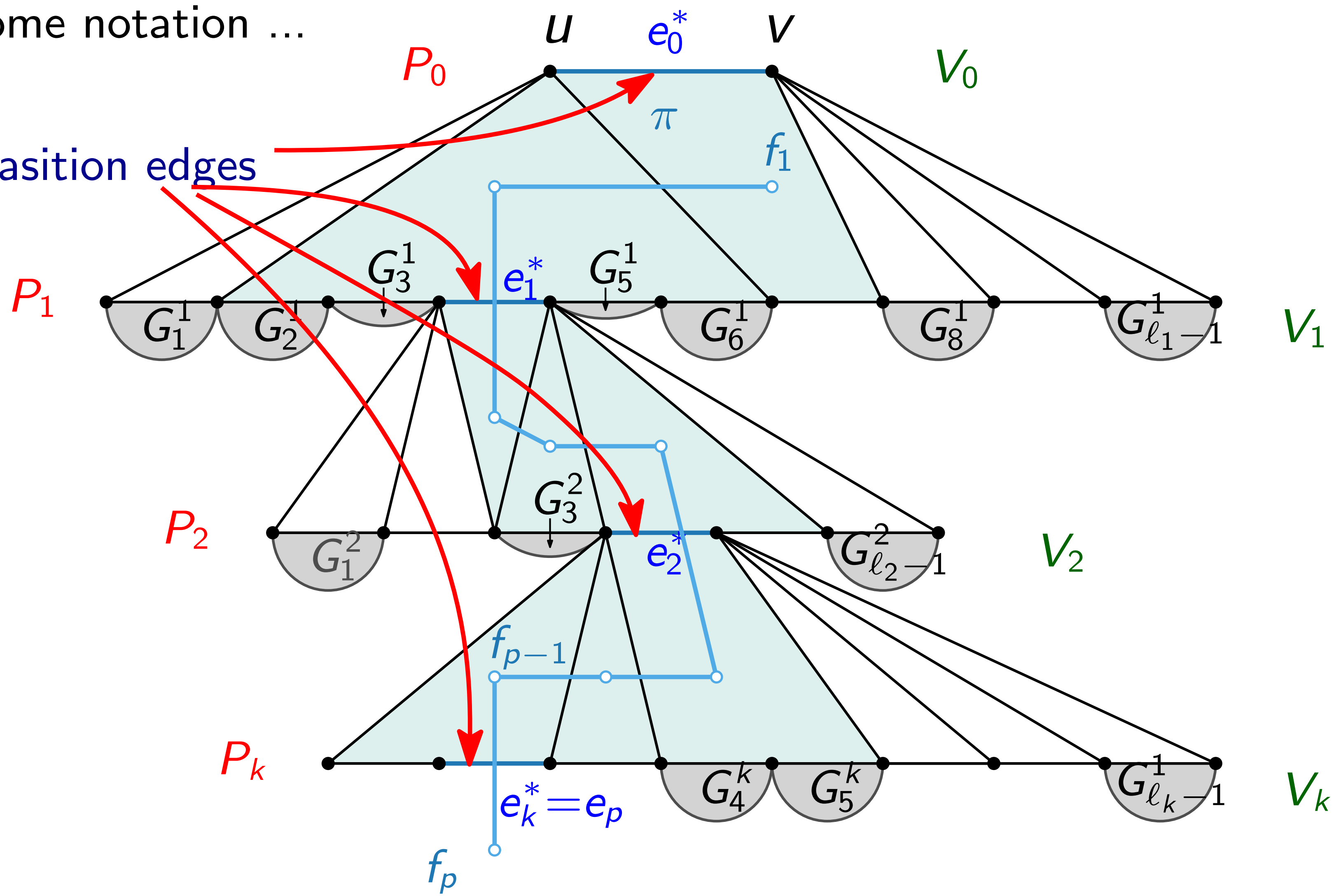

- $k \approx$ Number of transition edges

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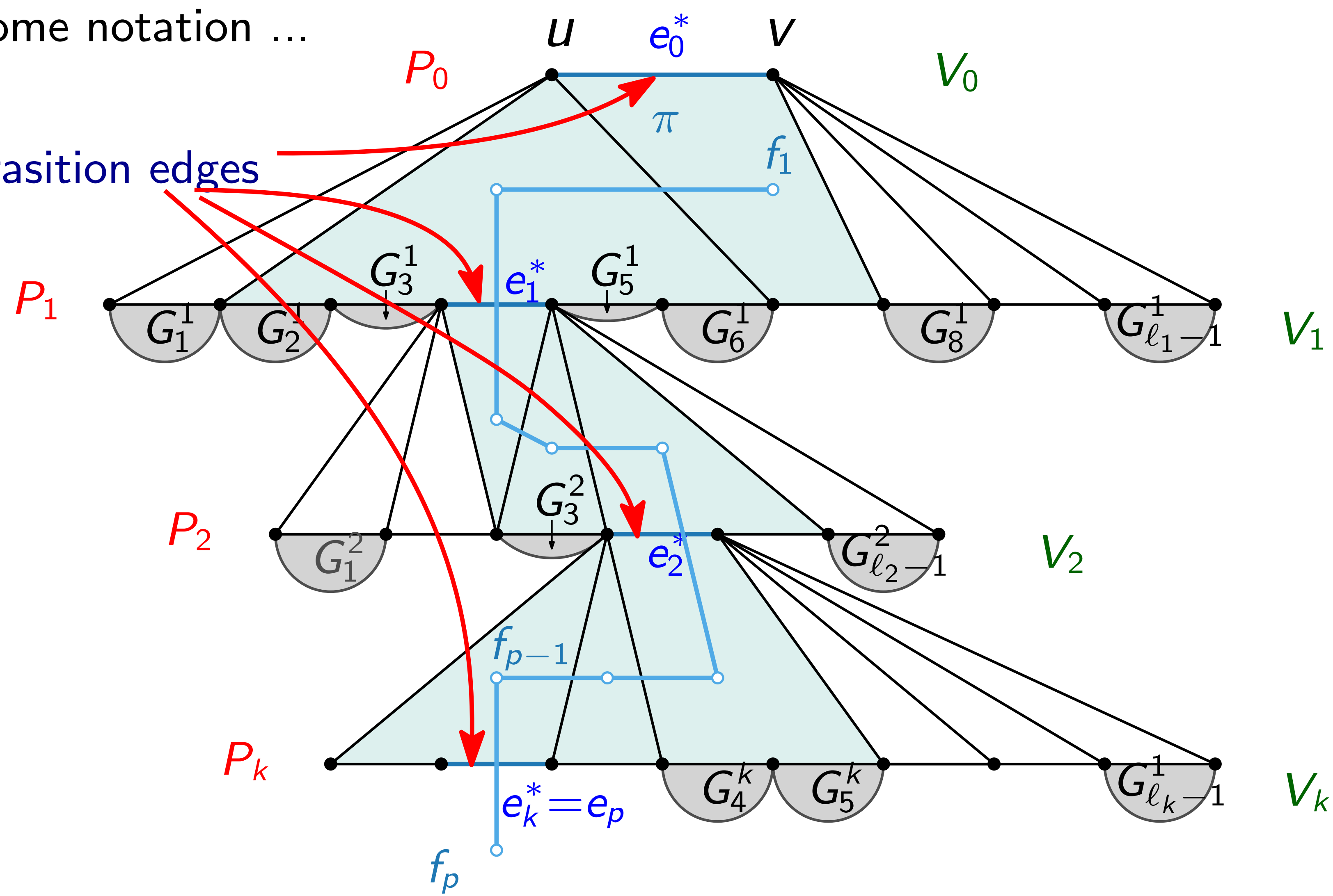

- Trasition edges

- $k \approx$ Number of transition edges

- We will distinguish cases on whether $k \leq \sqrt{n}$ or not.

Outerplanar graphs – Internally convex drawings

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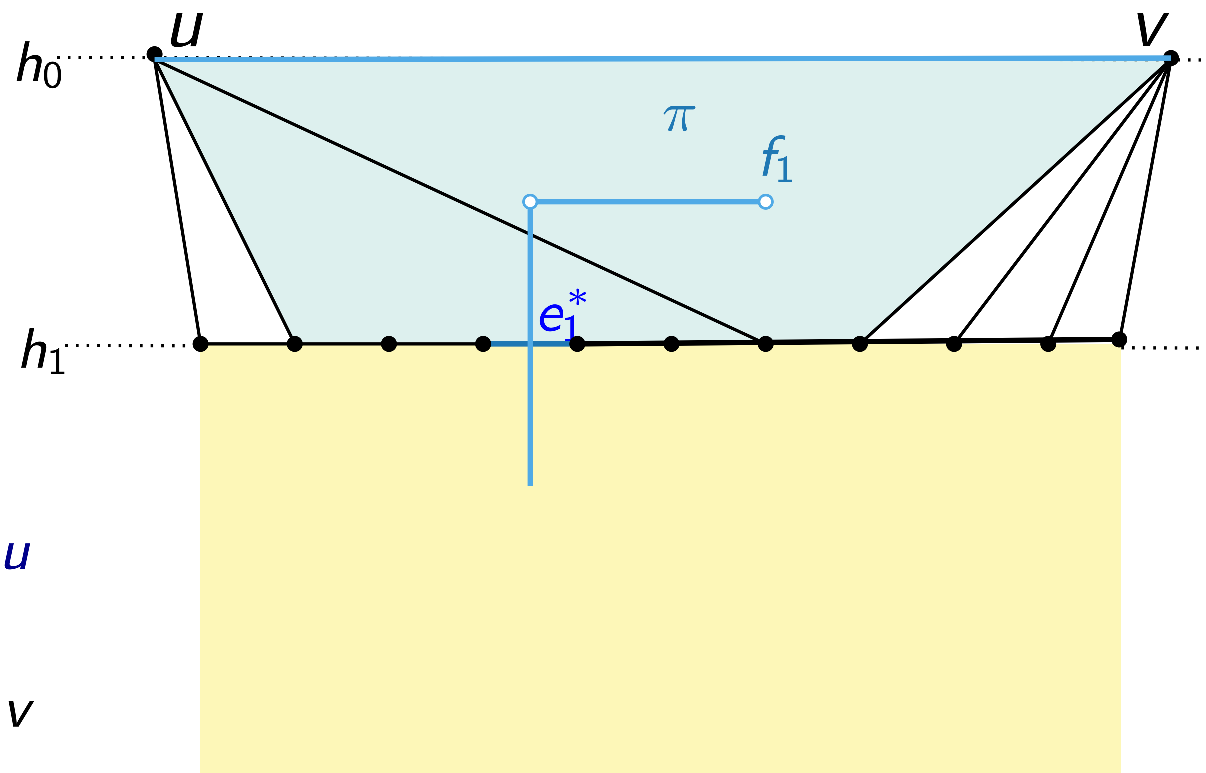
- *uv*-separated drawing

Outerplanar graphs – Internally convex drawings

- *uv*-separated drawing
  1. u and v lie on h_0
  2. All neighbors of u and v lie on h_1
  3. All other vertices lie on or below h_1
  4. All vertices different than u lie to its right
  5. All vertices different than v lie to its left



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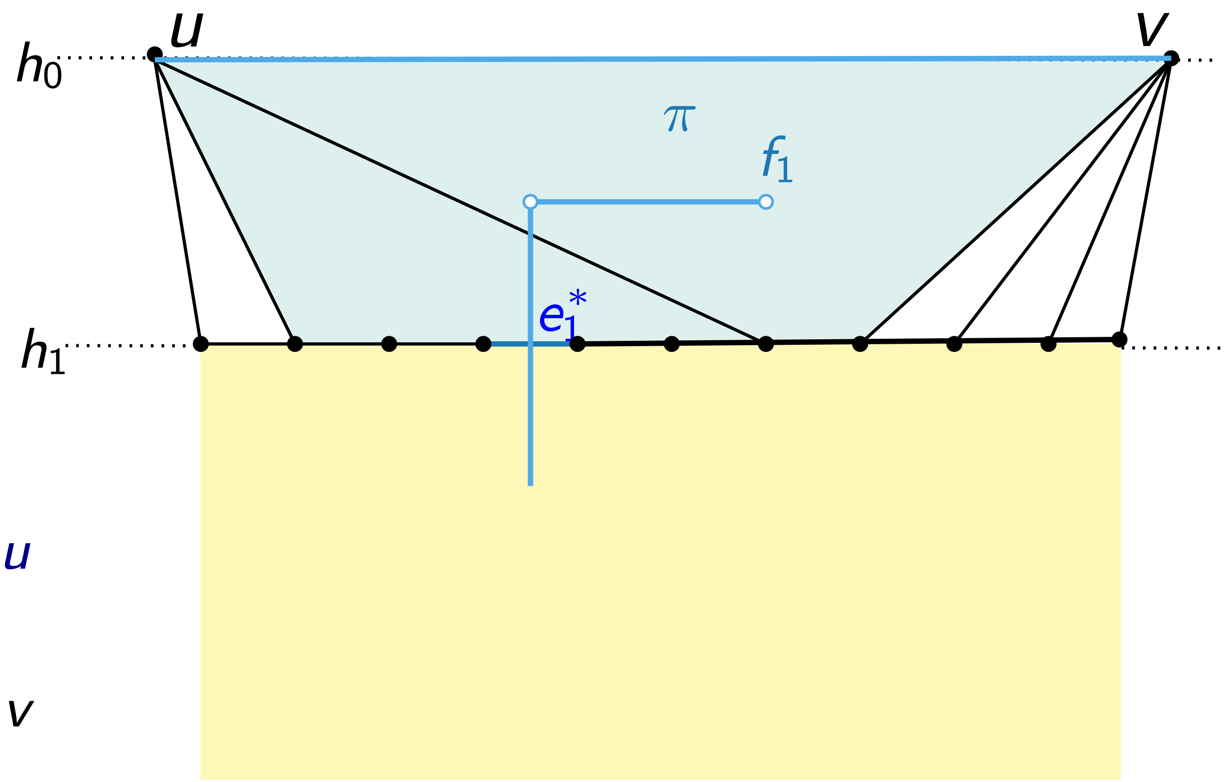

- Property-W: Every vertical line contains at least a vertex of G

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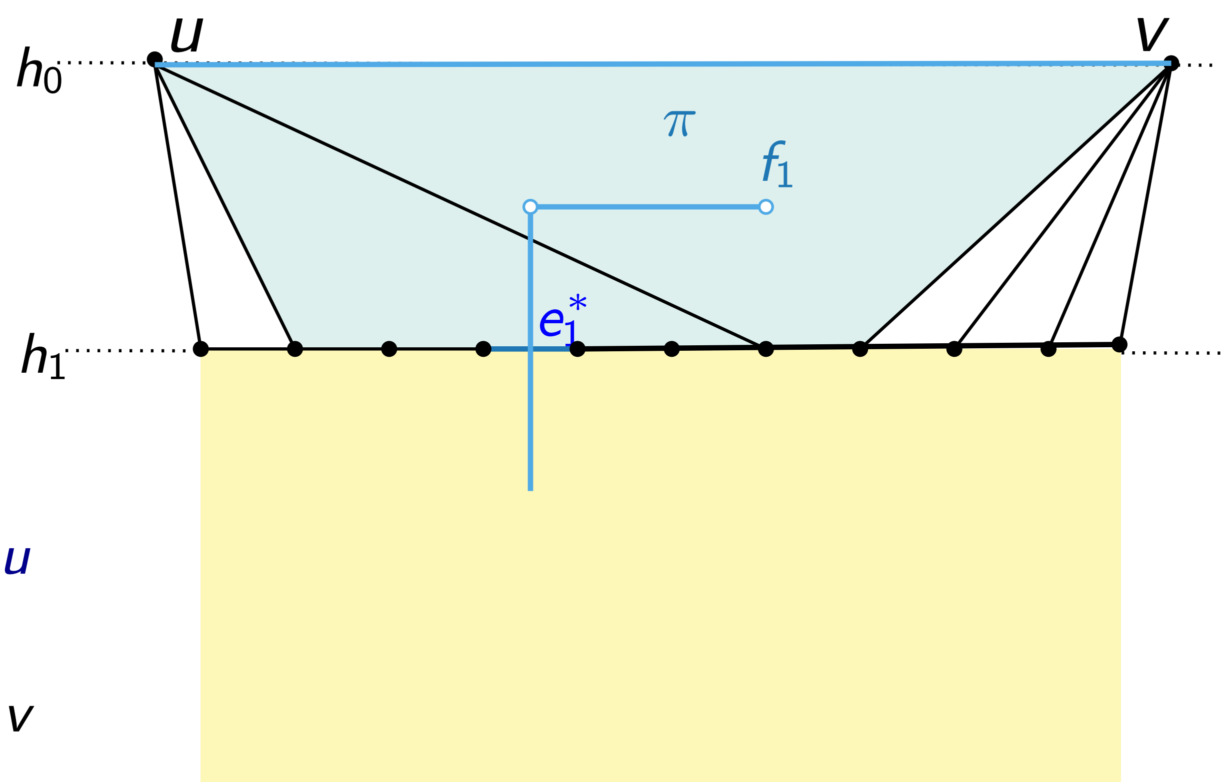

- Property-W: Every vertical line contains at least a vertex of G
- Property-H: Every horizontal line contains at least a vertex of G and the height of the drawing is at most $f(d)$

Outerplanar graphs – Internally convex drawings

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The Algorithm...

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- Step 1

Outerplanar graphs – Internally convex drawings

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  Find an outerpath based on thm BLLM-2024.

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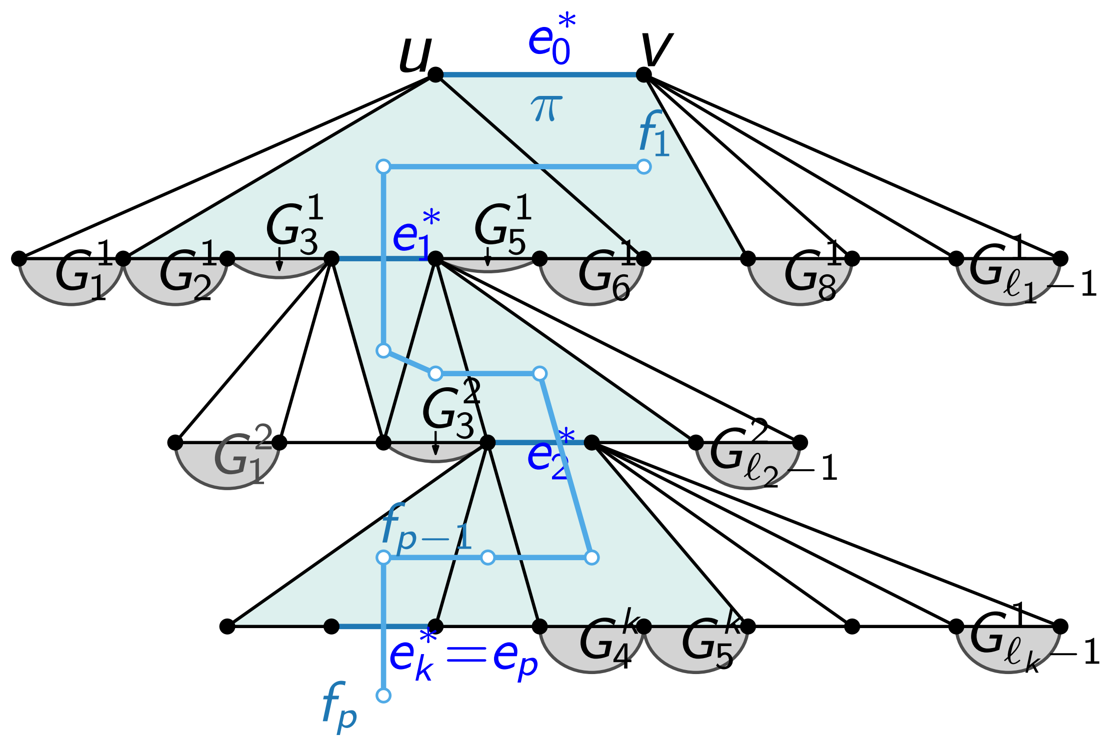

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- Step 2 .... Case 1: $k \leq \sqrt{n}$

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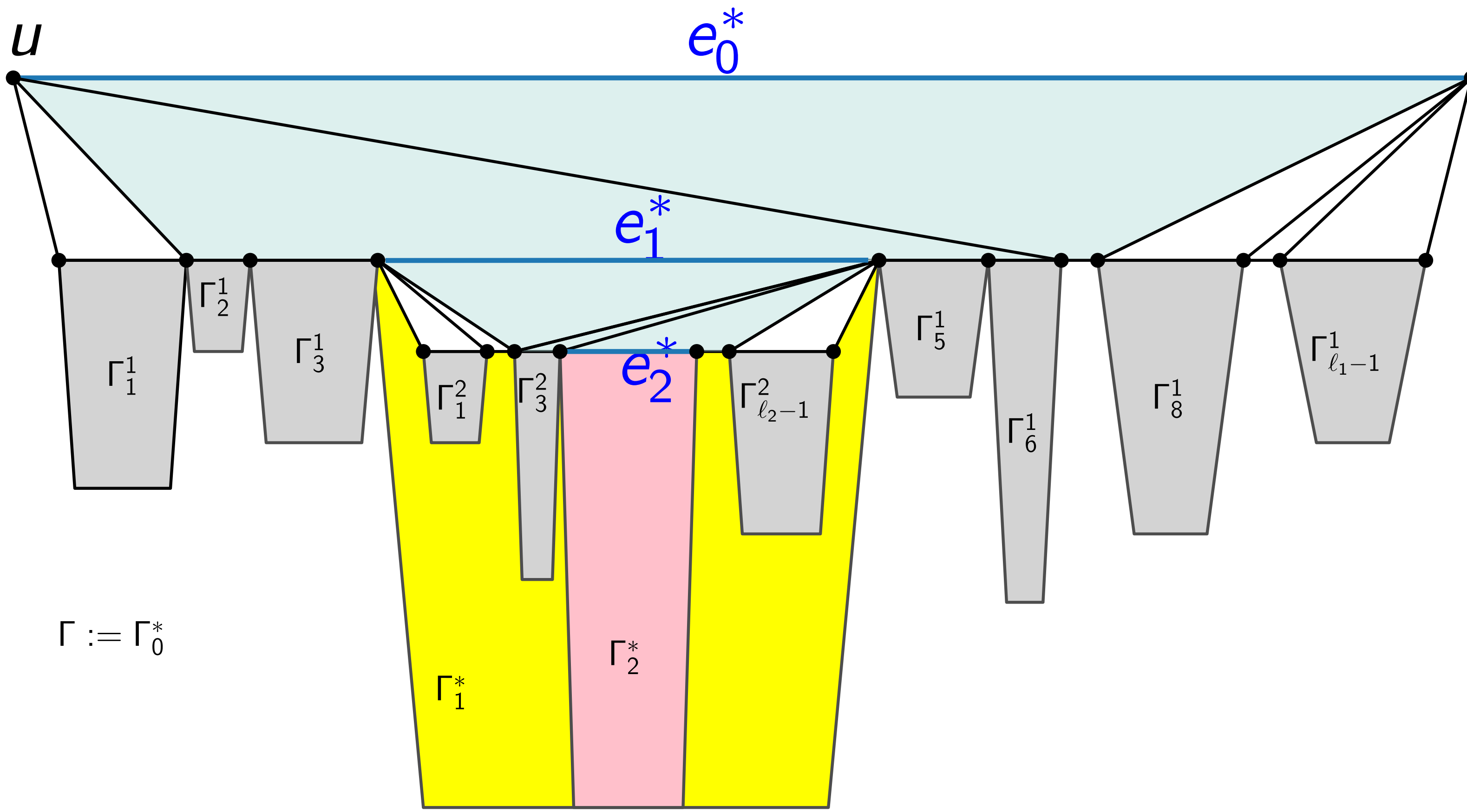

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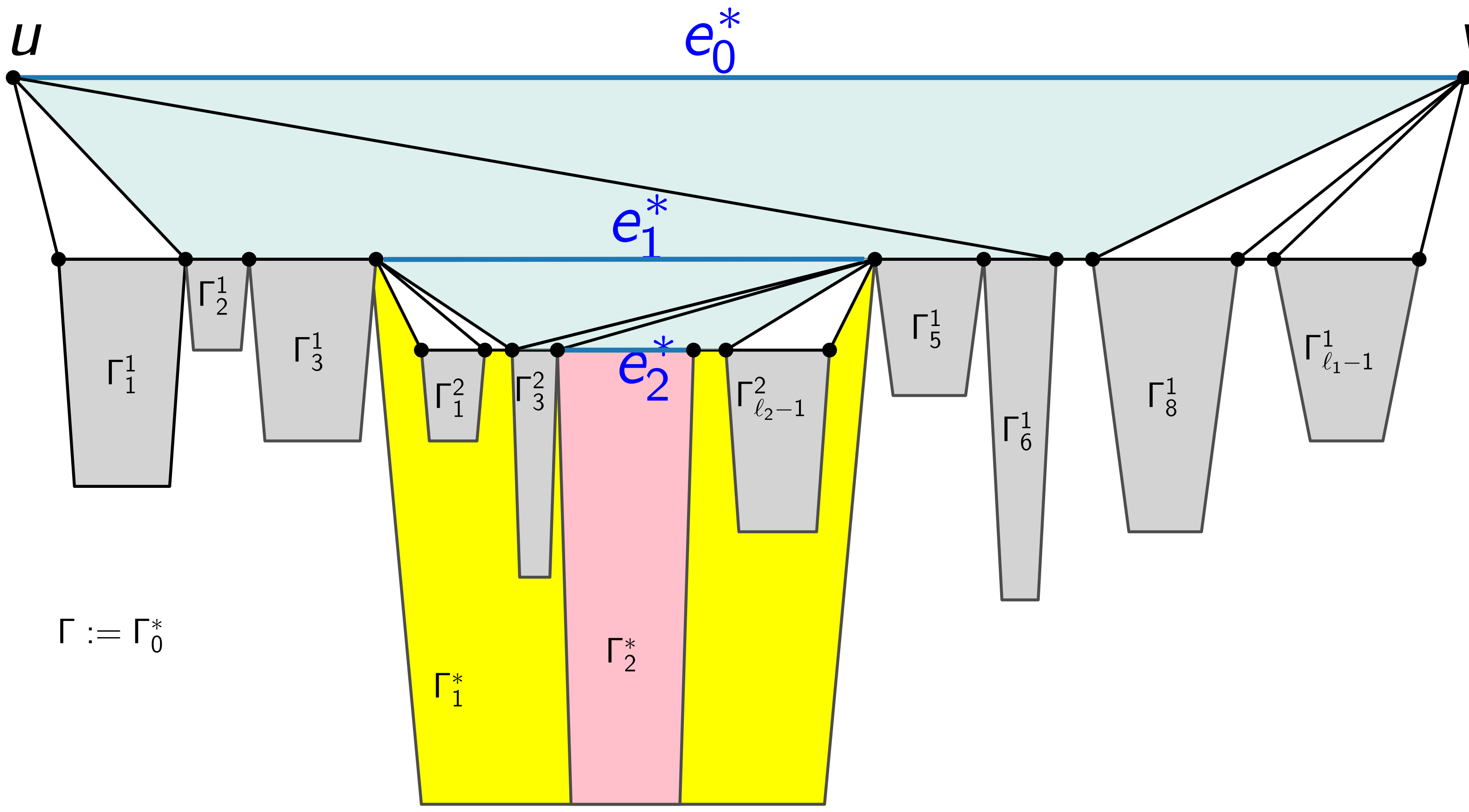

- Each sub-drawing satisfies:
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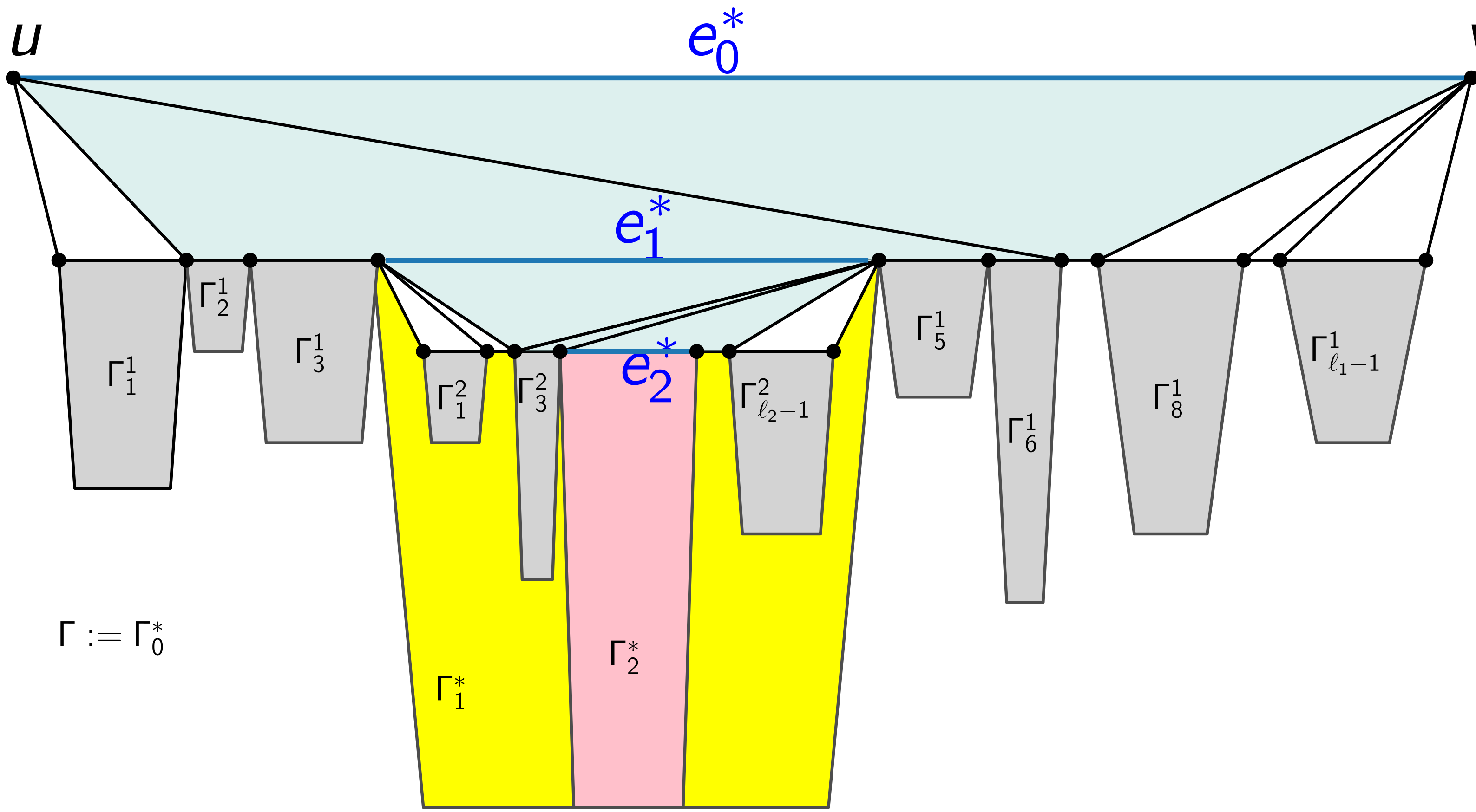

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- $O(n\sqrt{d}) = O(n\sqrt{n})$ area

Outerplanar graphs – Internally convex drawings

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- Step 2 .... Case 2: $k > \sqrt{n}$

Outerplanar graphs – Internally convex drawings

The Algorithm...

- Step 2 .... Case 2: $k > \sqrt{n}$

Lemma

There exists an index t with $2 \leq t \leq 1 + \sqrt{n}$ such that $|V_t| \leq \sqrt{n}$.

Outerplanar graphs – Internally convex drawings

2.a Choose t such that $|V_t| \leq \sqrt{n}$

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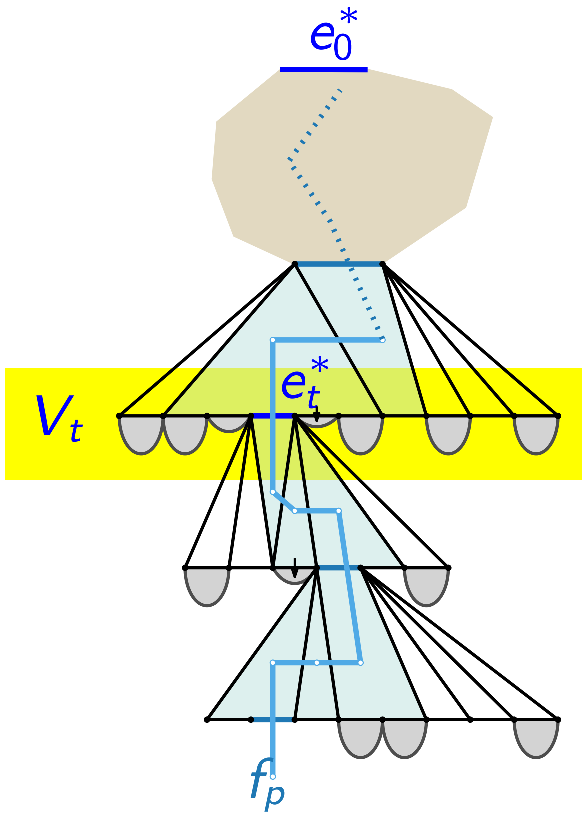

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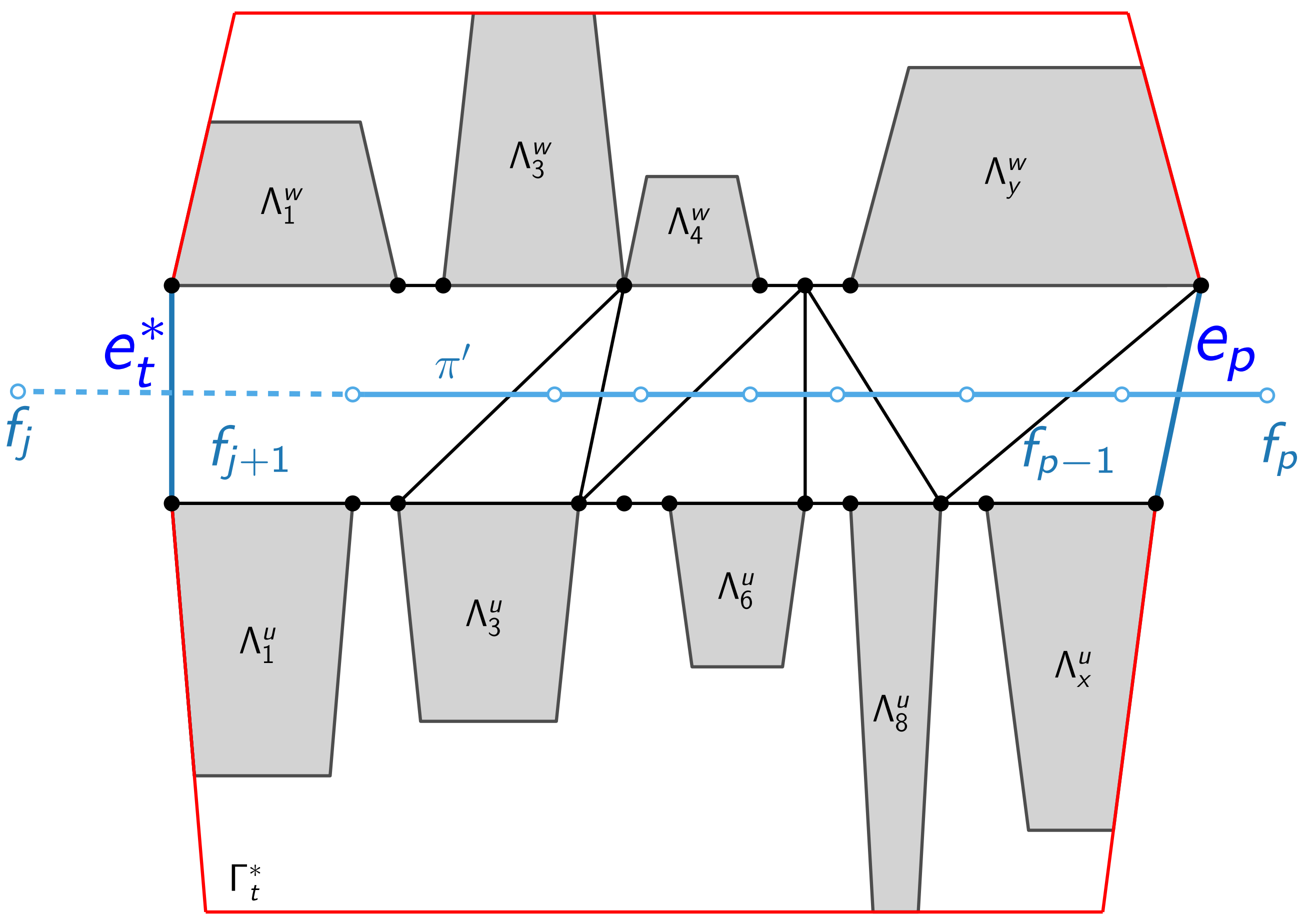

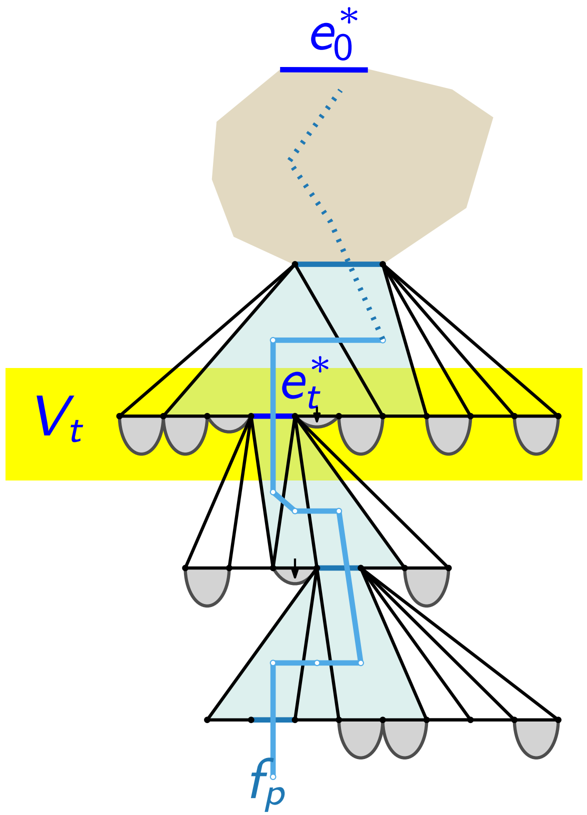

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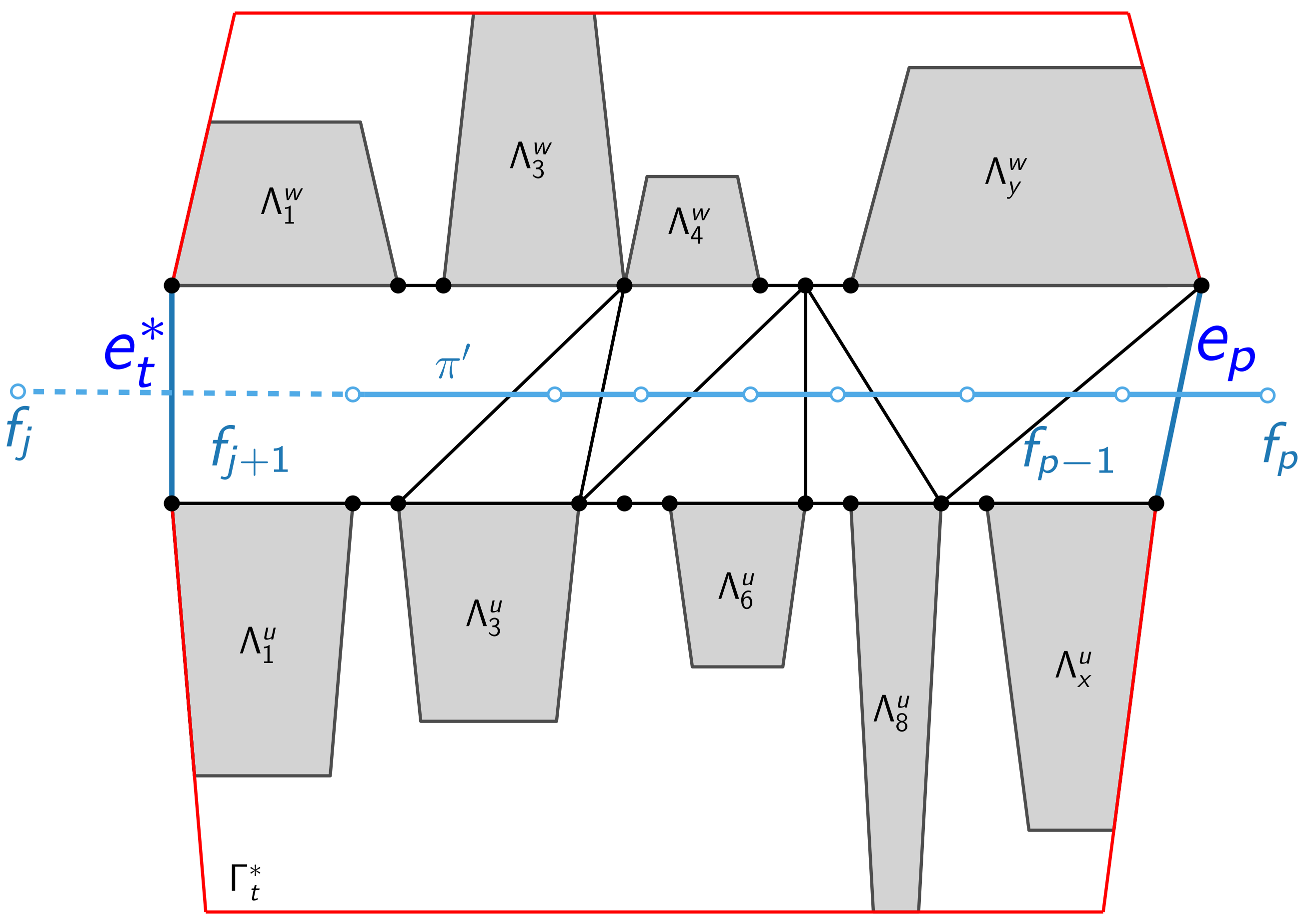

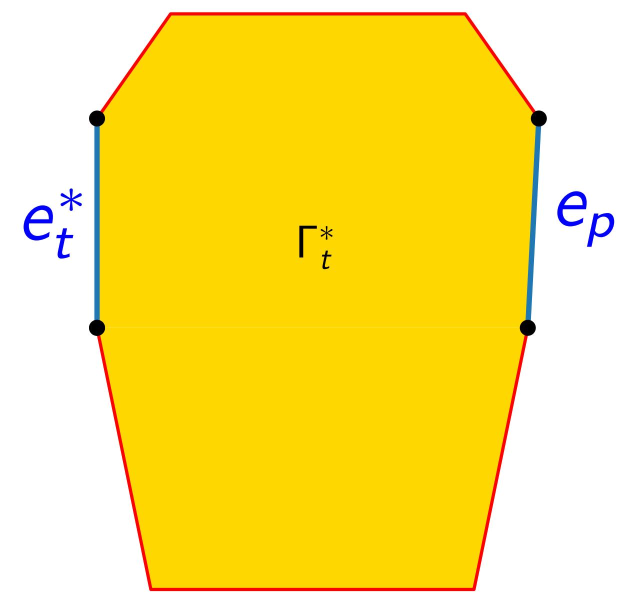

Outerplanar graphs – Internally convex drawings

2.c Augment the drawing of G_t^* .... to a drawing of G_{t-1}^*

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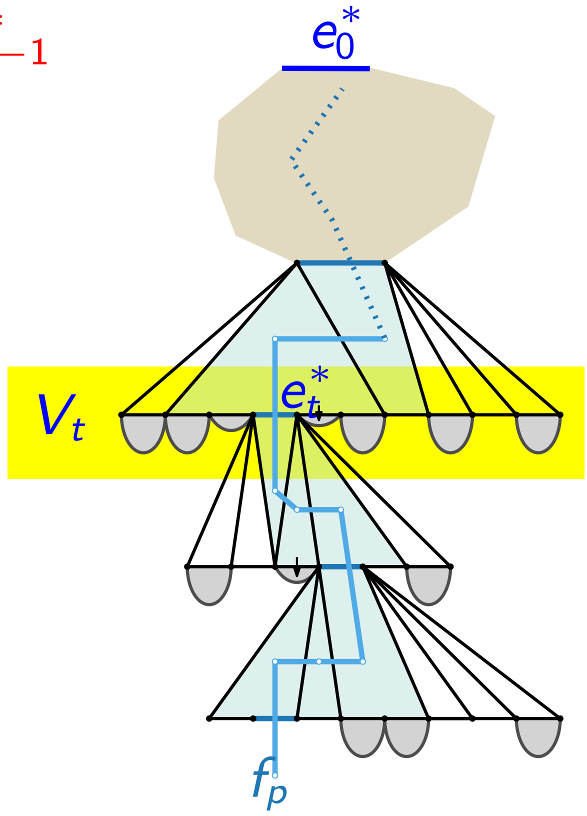

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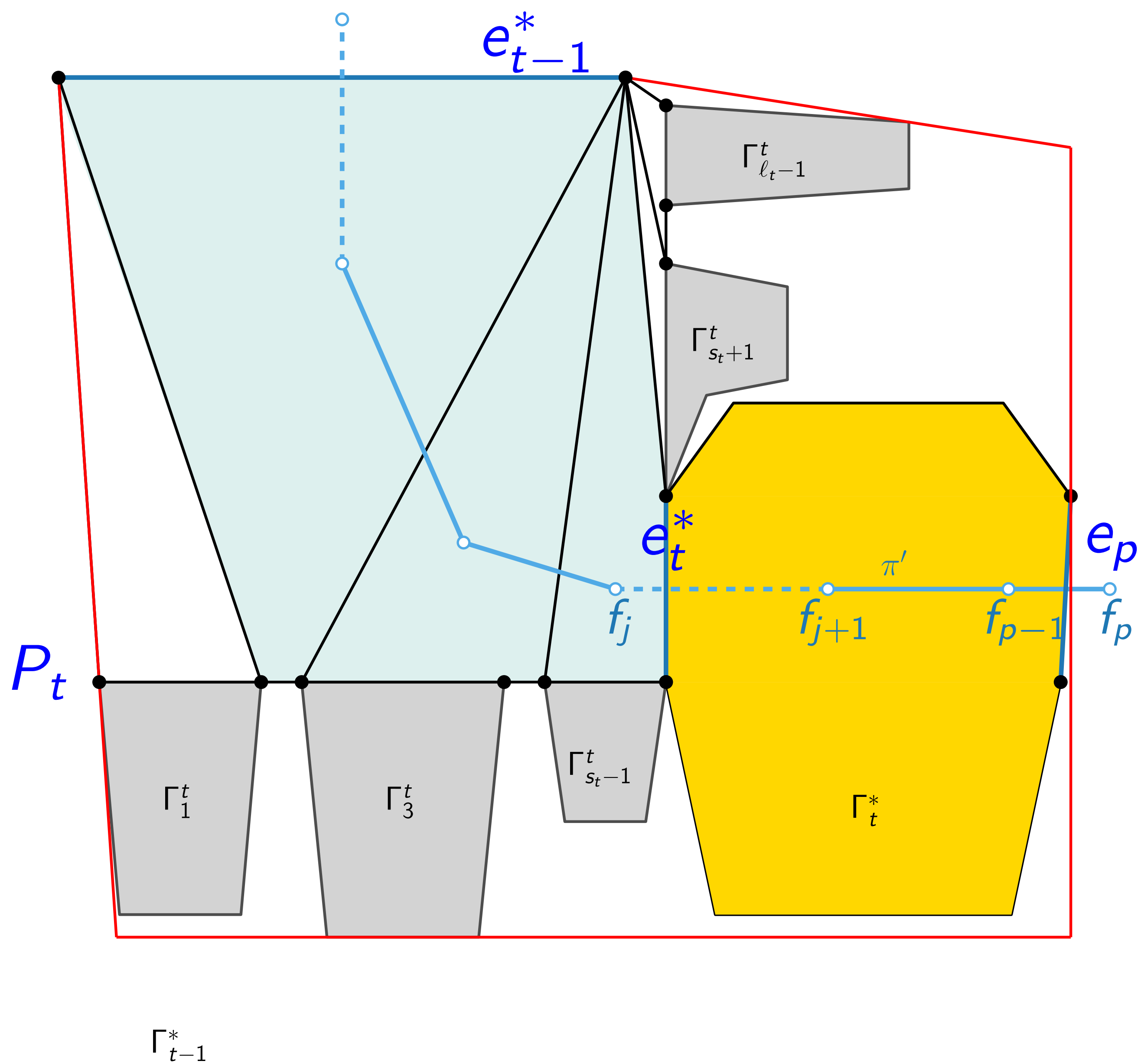

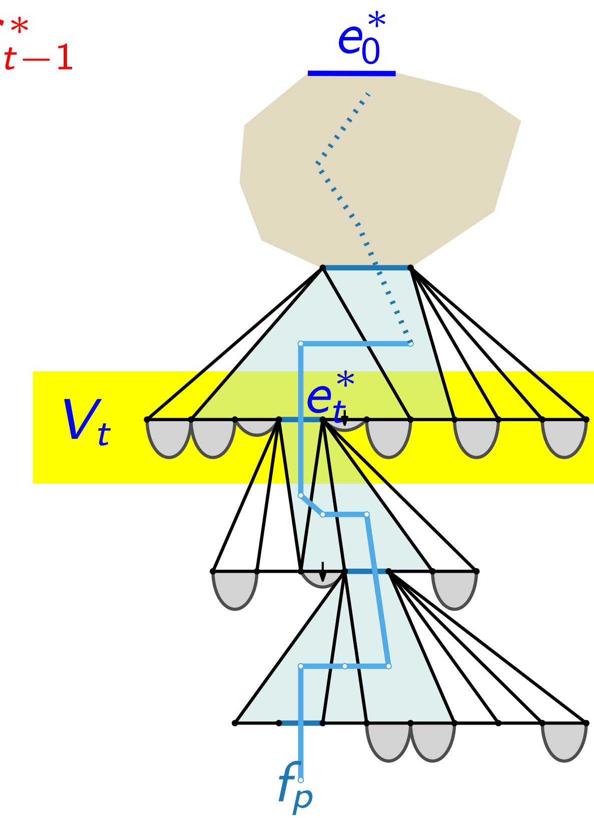

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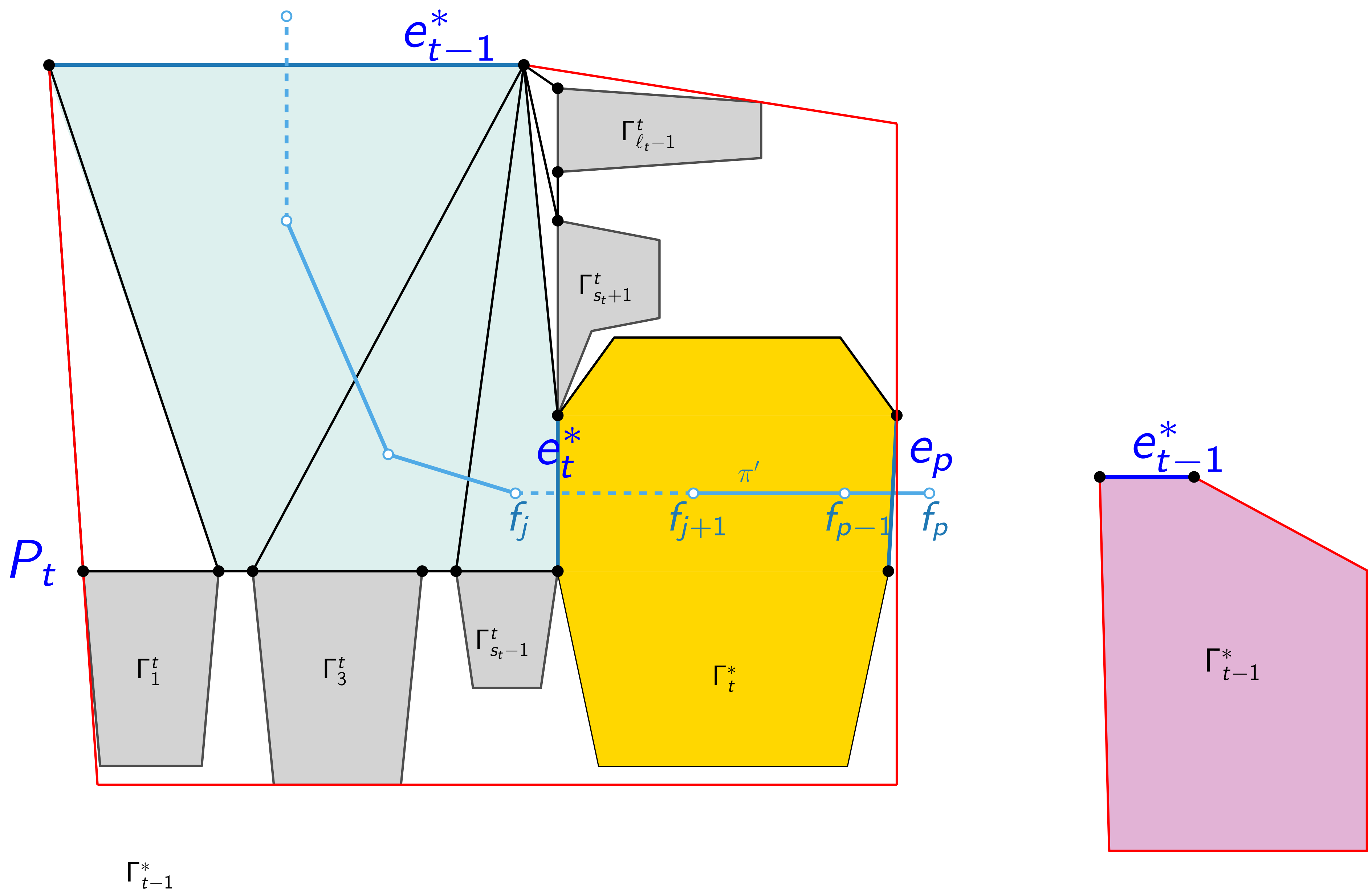

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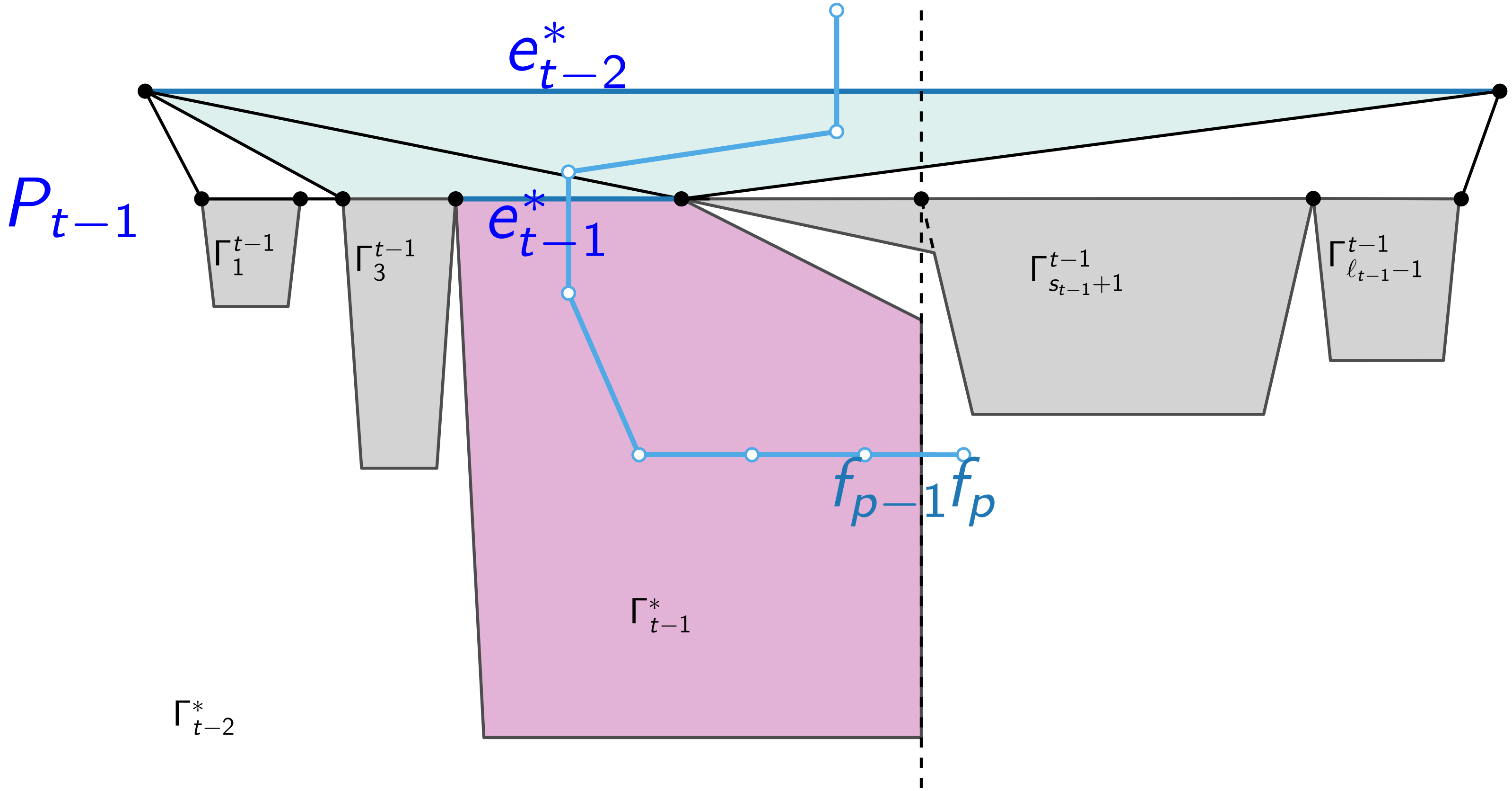

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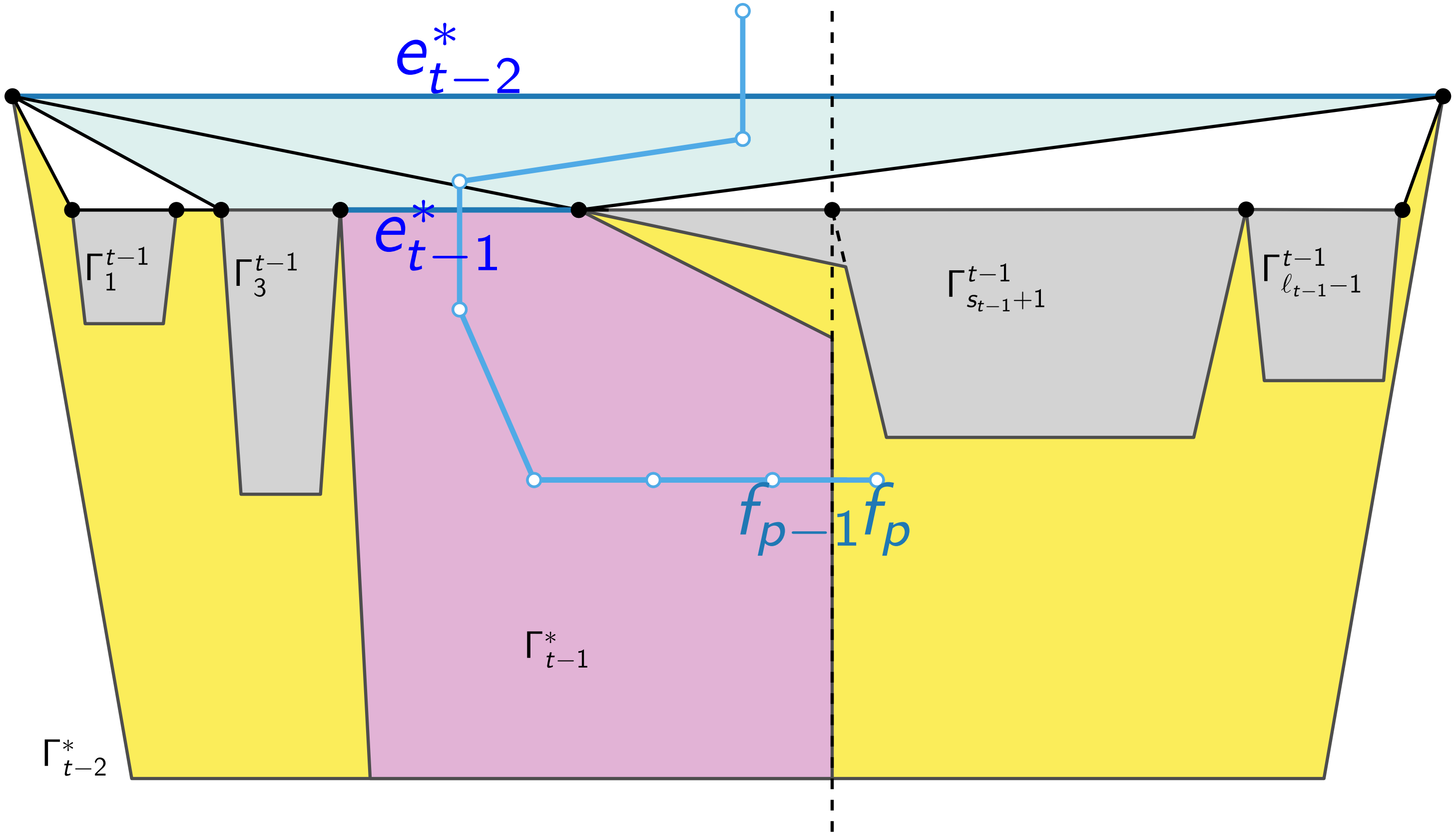

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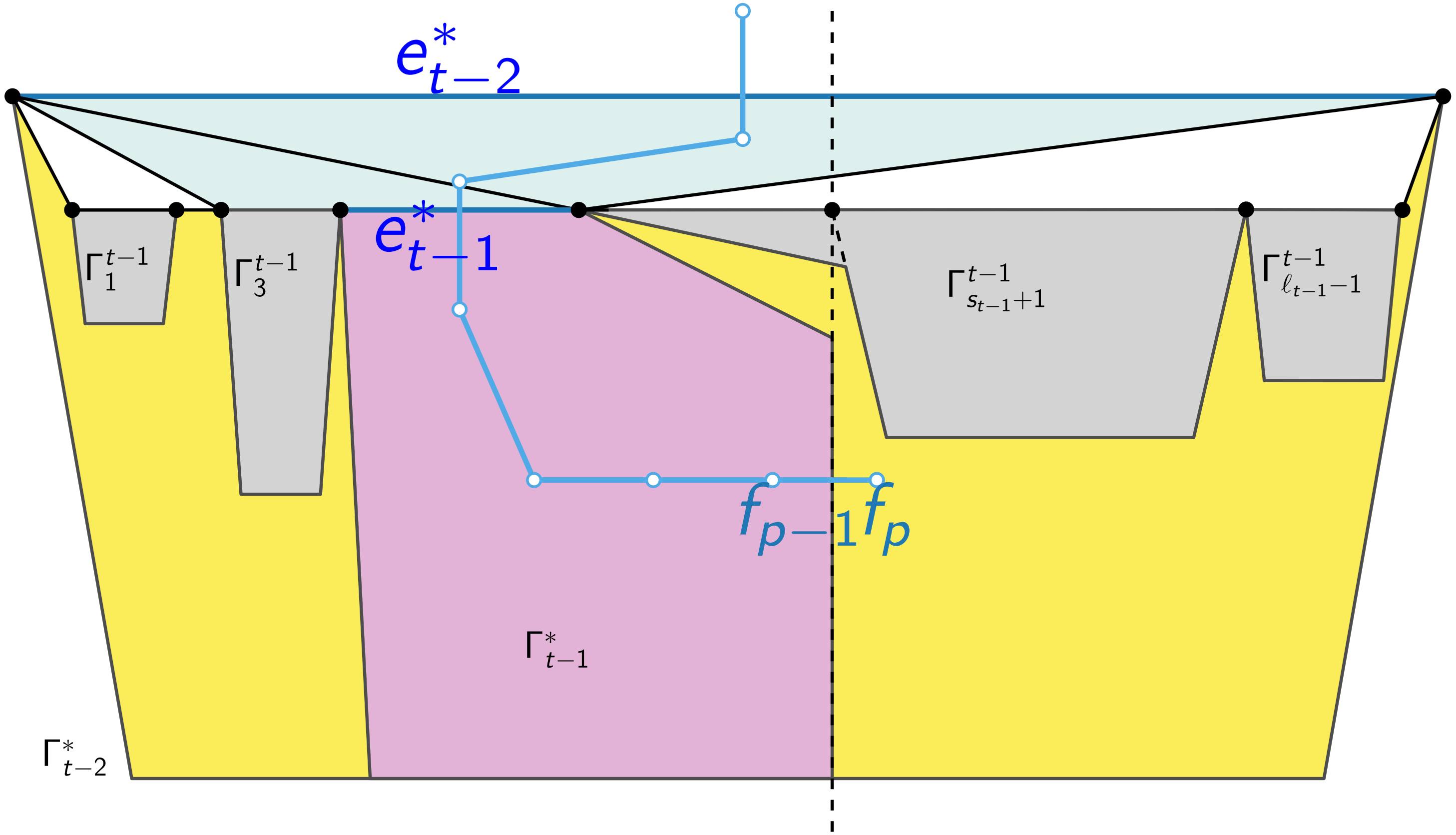
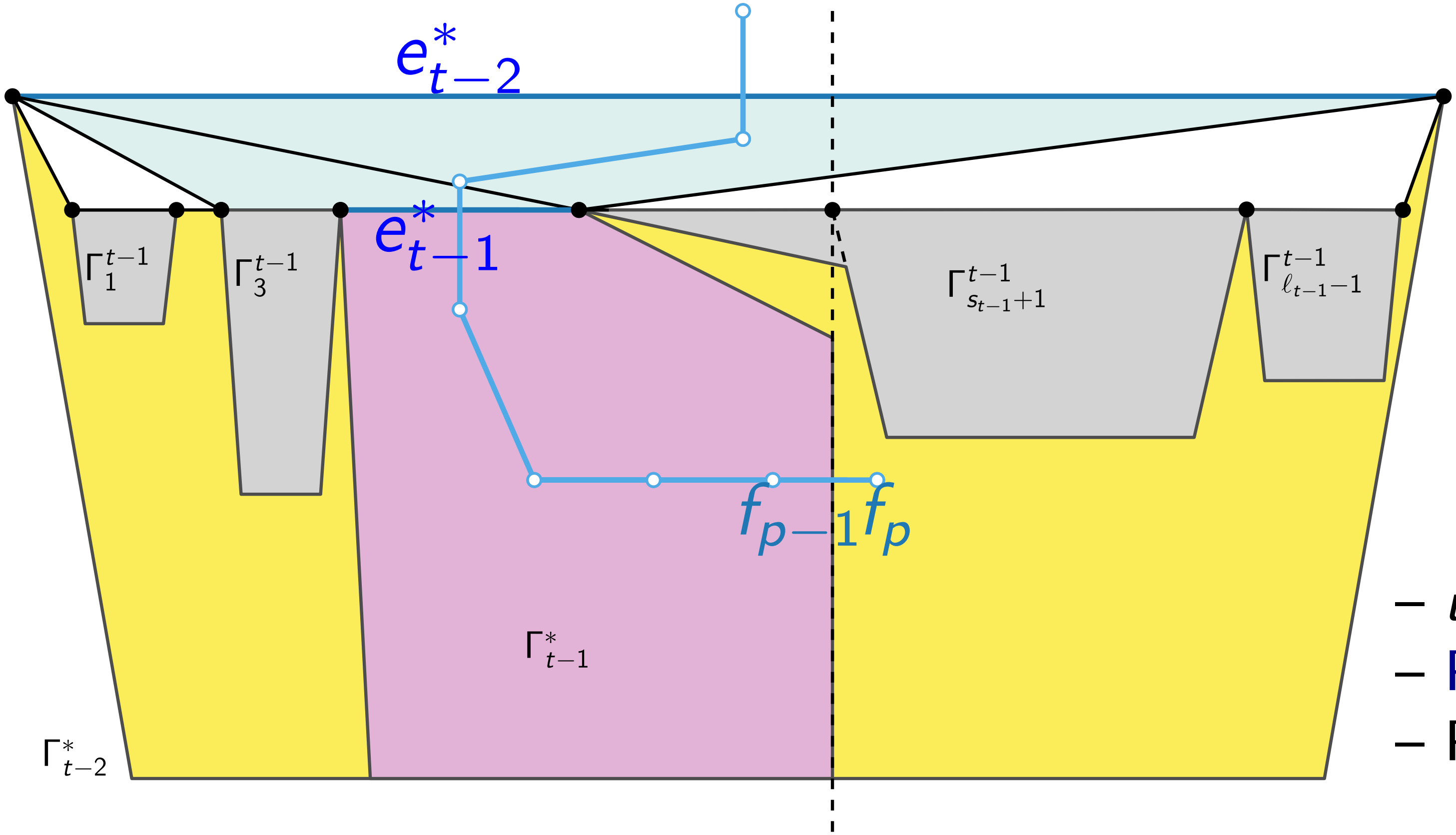


2.e Continue iteratively to build the drawing of G_0 [as in case 1]

Outerplanar graphs – Internally convex drawings

2.d Augment the drawing of G^*_{t-1} .... to a drawing of G^*_{t-2}



– *uv*-separated
– Property-W
– Property-H

2.e Continue iteratively to build the drawing of G_0 [as in case 1]

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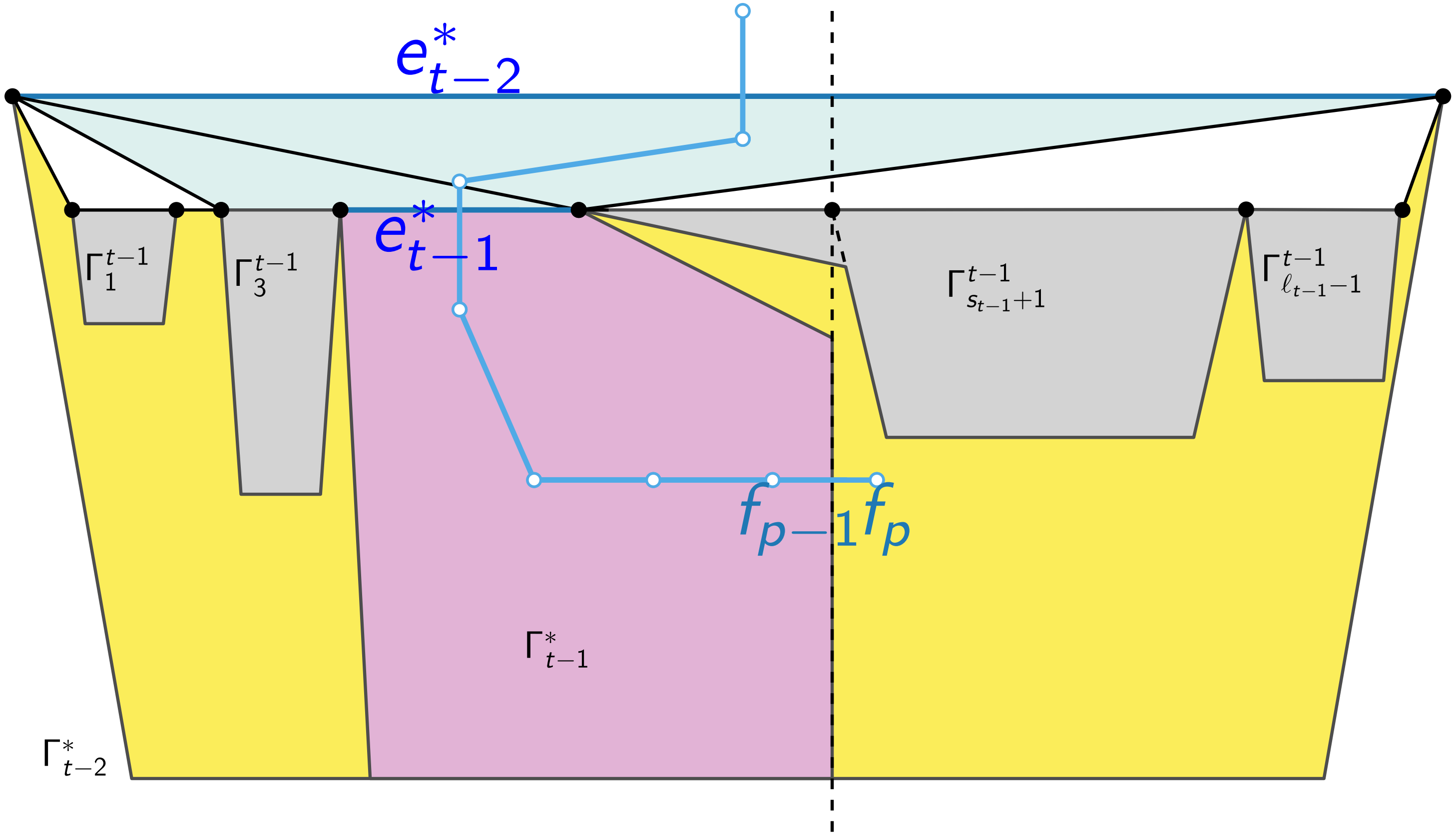

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Theorem

Every n-vertex outerplane graph admits an embedding-preserving internally-convex grid drawing in $O(n^{1.5})$ area

Conclusion

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Our results:

- Outerplanar graphs: $O(n^{1.5})$ area internally convex drawing
- Outerpaths: $O(nk^2)$ area internally strictly convex drawing.

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- Close the gap between the $\Omega(n)$ lower bound and the $O(n^{1.5})$ upper bound for internally convex drawing of outerplanar graphs.

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- Can we build $O(n^2)$-area internally strictly convex drawings for outerplanar graphs with internal faces of size at most 4?