

# The PAGENUMBER of monotone DIRECTED acyclic outerplanar graphs is 4 or 5

Michael Kaufmann

with Jawaherul Alam, Michalis Bekos, Martin Gronemann

GD 2025

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# Previous work on linear orders

- Michael A. Bekos, MK, Fabian Klute, Sergey Pupyrev, Chrysanthi N. Raftopoulou, Torsten Ueckerdt: Four Pages Are Indeed Necessary for Planar Graphs. J. Comput. Geom. 11(1): 332-353 (2020)
- Jawaherul Md. Alam, Michael A. Bekos, Martin Gronemann, MK, Sergey Pupyrev: Queue Layouts of Planar 3-Trees. Algorithmica 82(9): 2564-2585 (2020)
- Md. Jawaherul Alam, Michael A. Bekos, Vida Dujmovic, Martin Gronemann, MK, Sergey Pupyrev: On dispersable book embeddings. Theor. Comput. Sci. 861: 1-22 (2021)
- Jawaherul Md. Alam, Michael A. Bekos, Martin Gronemann, MK, Sergey Pupyrev: The mixed page number of graphs. Theor. Comput. Sci. 931: 131-141 (2022)

# The story on directed graphs.....

Not all directed planar graphs have boundedpagenumber

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Outerplanar graphs have bounded pagenumber  
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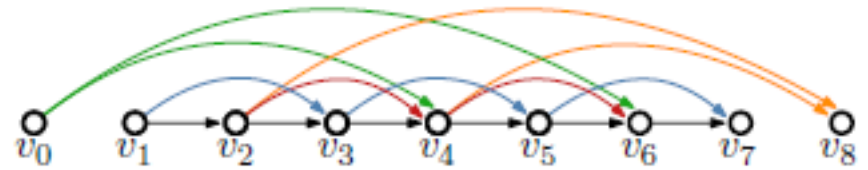
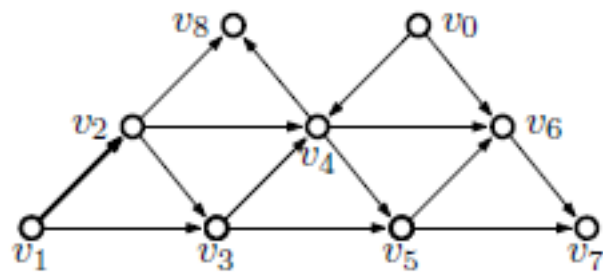
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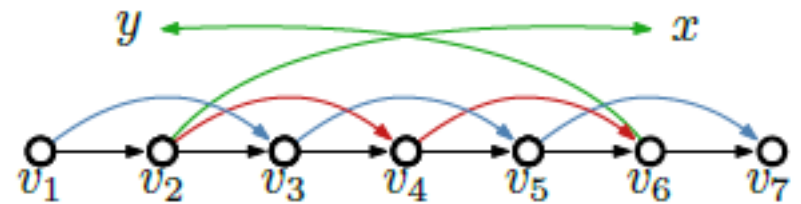
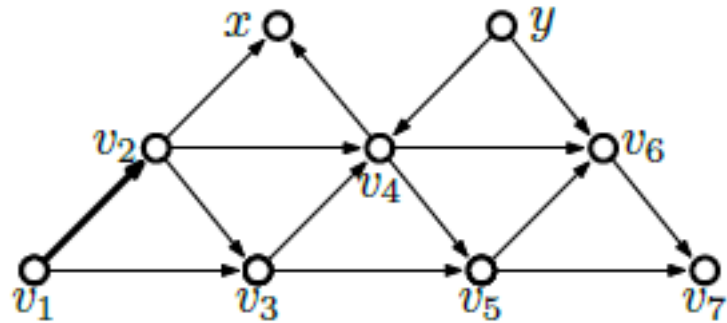
$s \leq 128$  (Nöllenburg/Pupyrev)

what's 'monotone'?

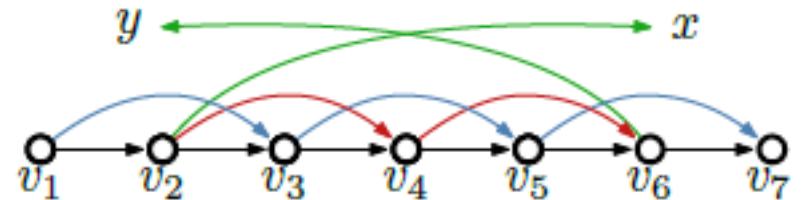
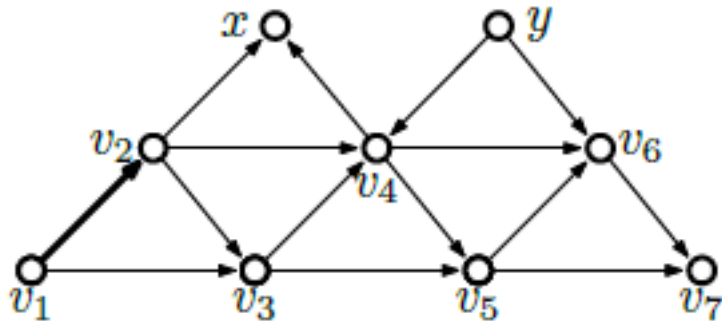
# Monotone outerplanar graphs



# Easy lower bound: 3 pages



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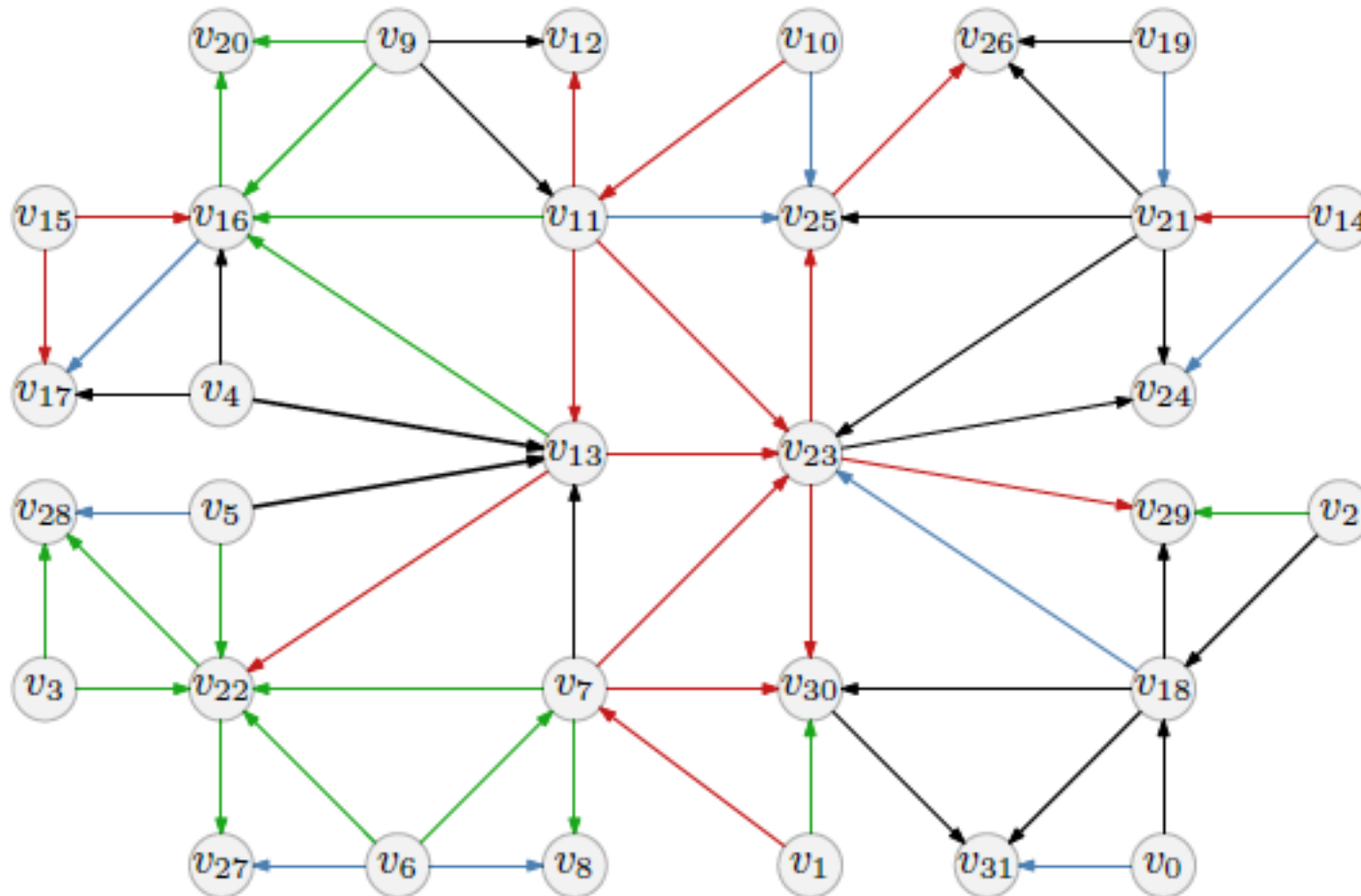


Assume  $G$  can be done in 2 pages.

$y$  is left of  $v_4$ ,  $x$  is right of  $v_4$

$(y, v_6)$  and  $(v_2, x)$  are both not red, but they cross !

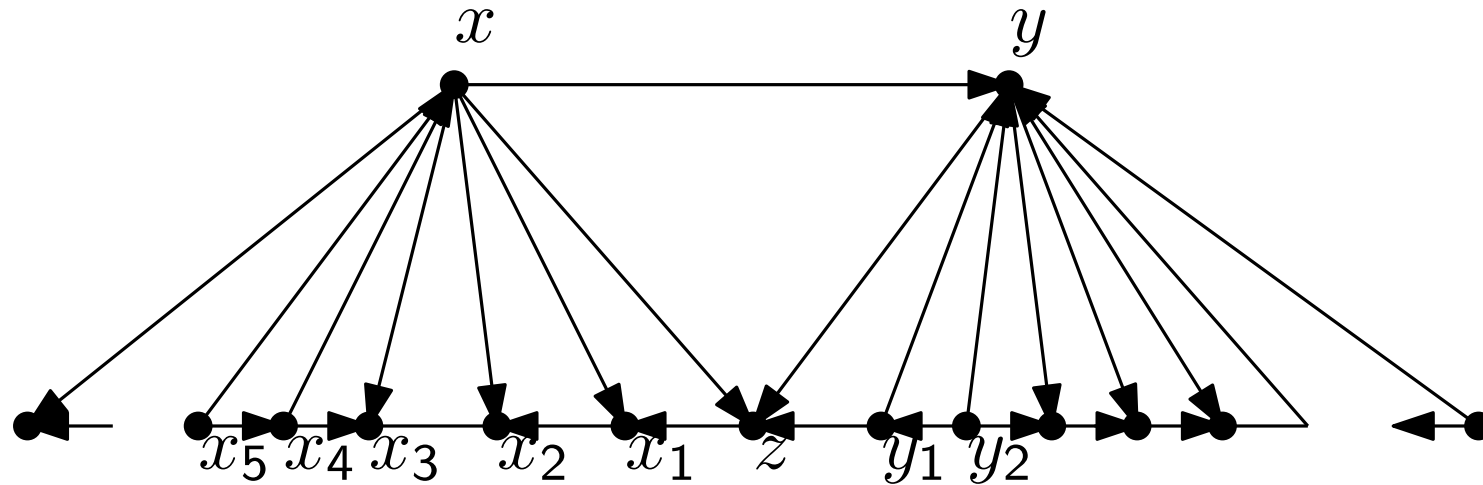
# Involved lower bound: 4



Proof by SAT formulation and –solving

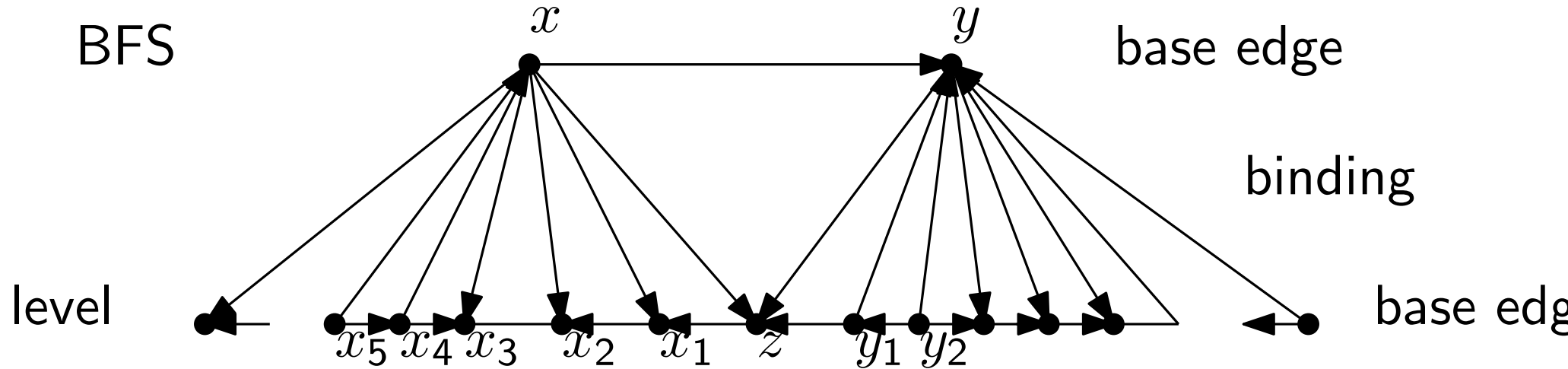
# A simple algorithm for 6 pages

Notation: - binding, level, critical, local



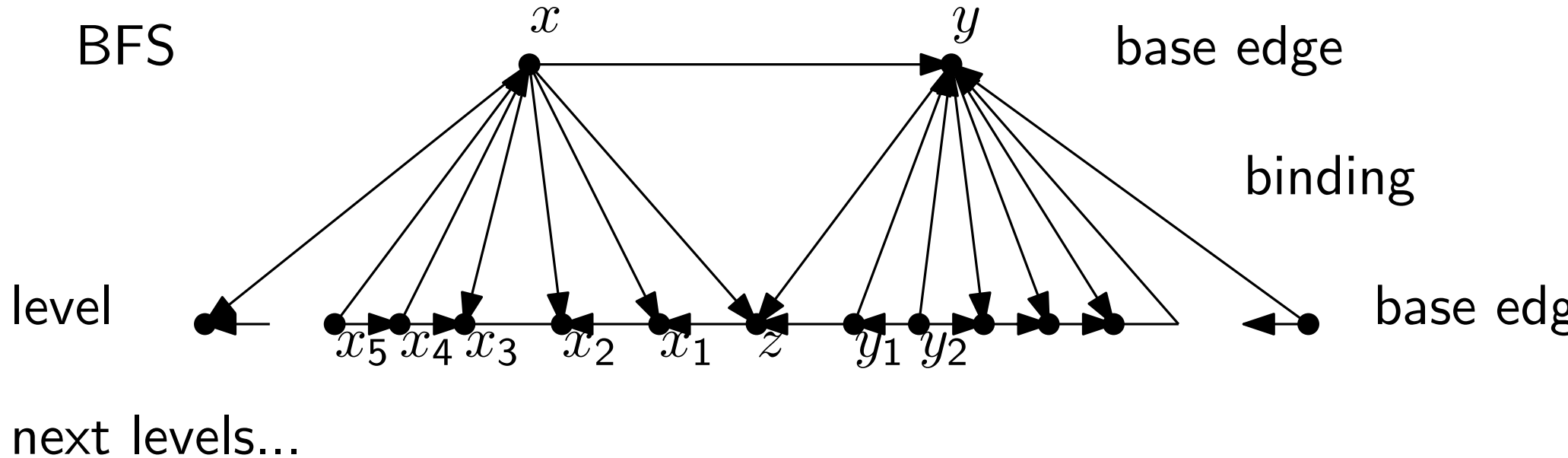
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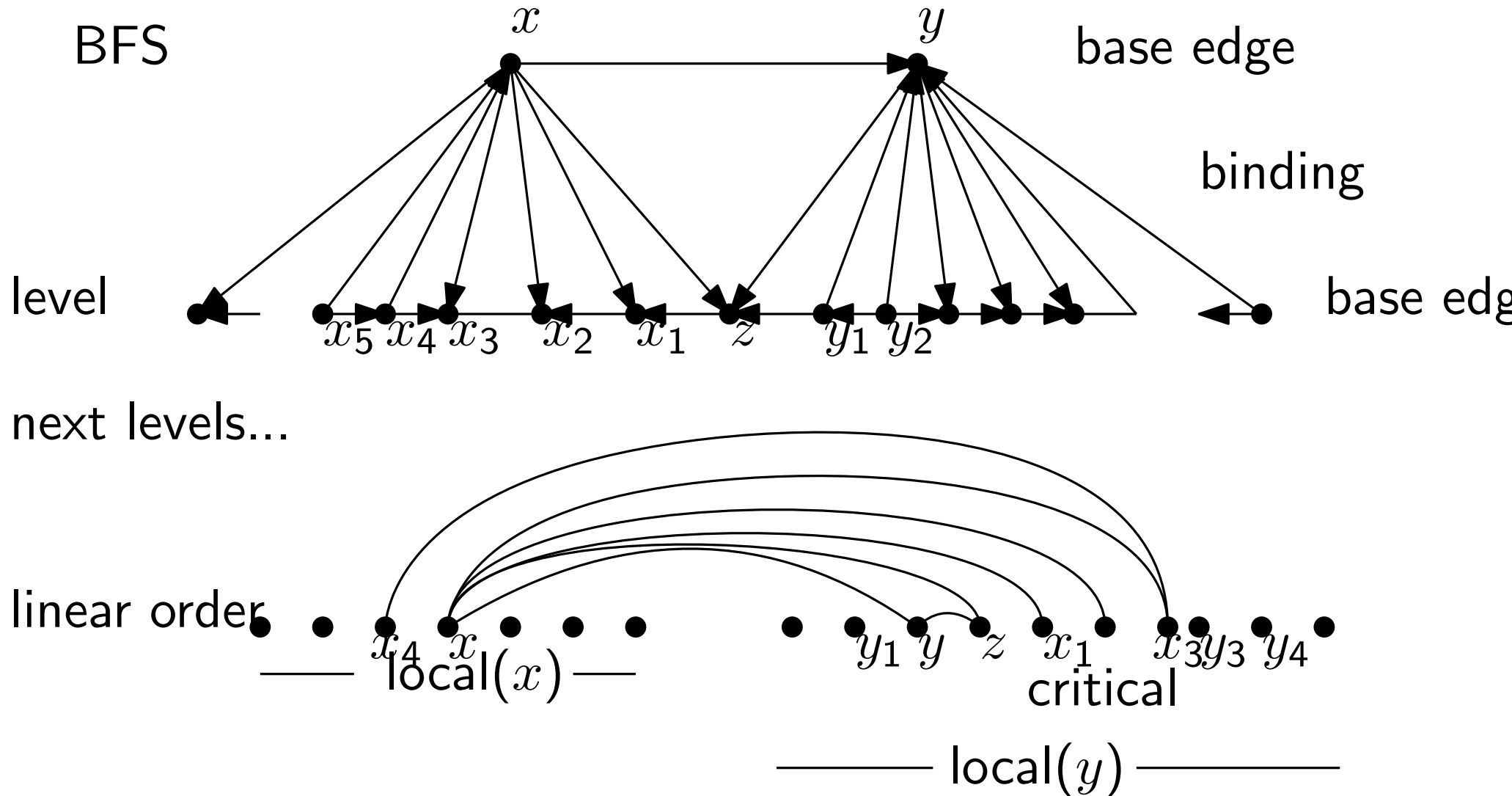
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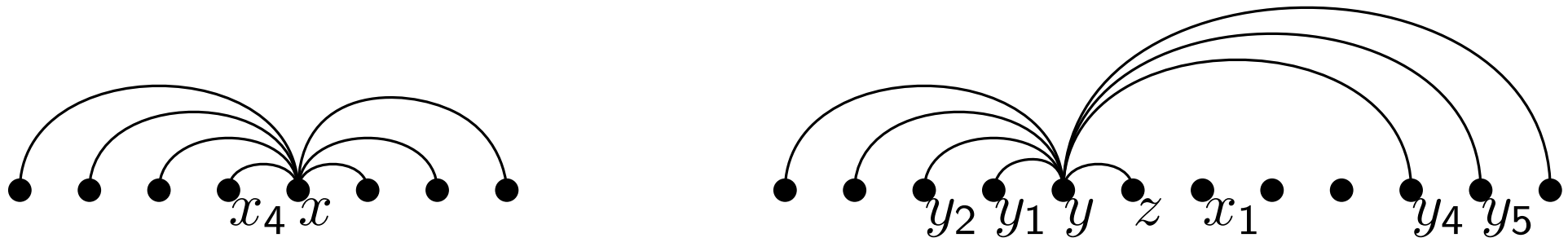
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## In more detail:

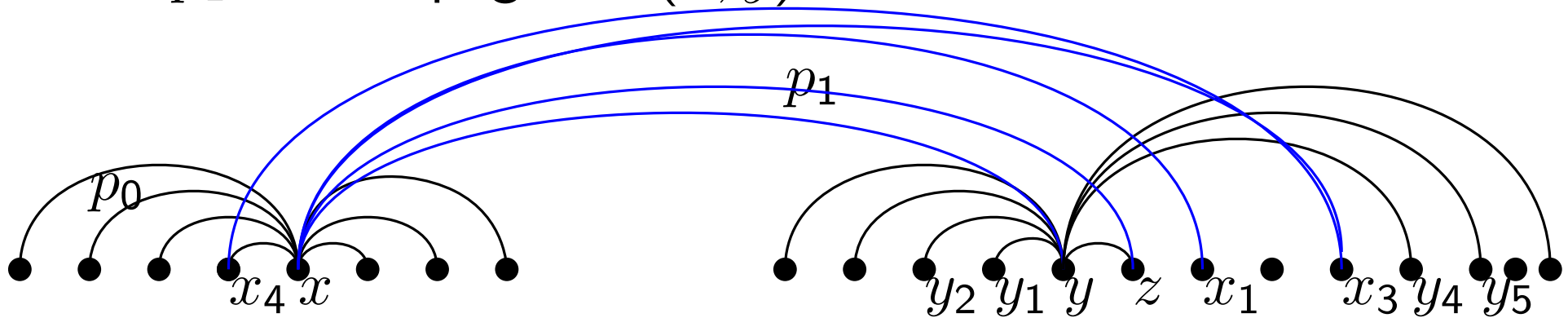
let  $p_1$  be the page for  $(x, y)$



- local binding edges from  $x$  and  $y$  go to  $p_0$

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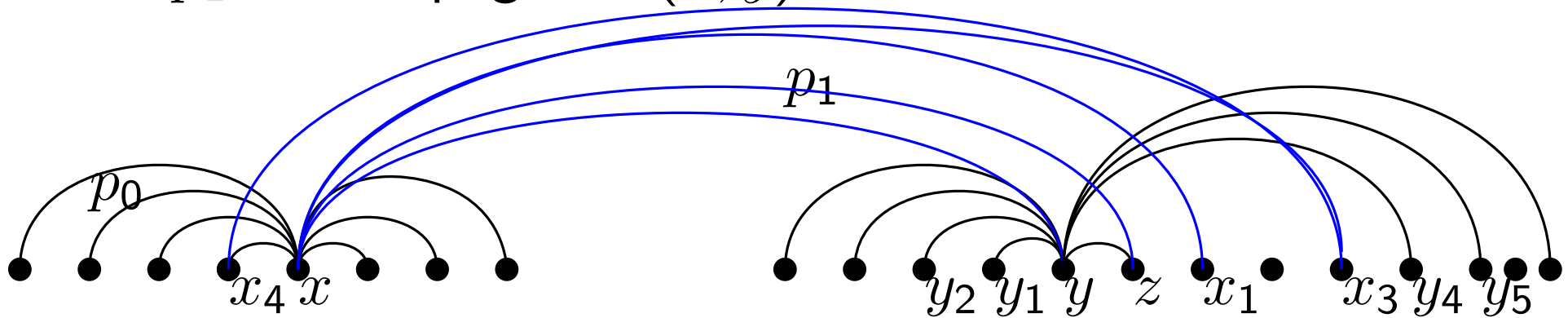
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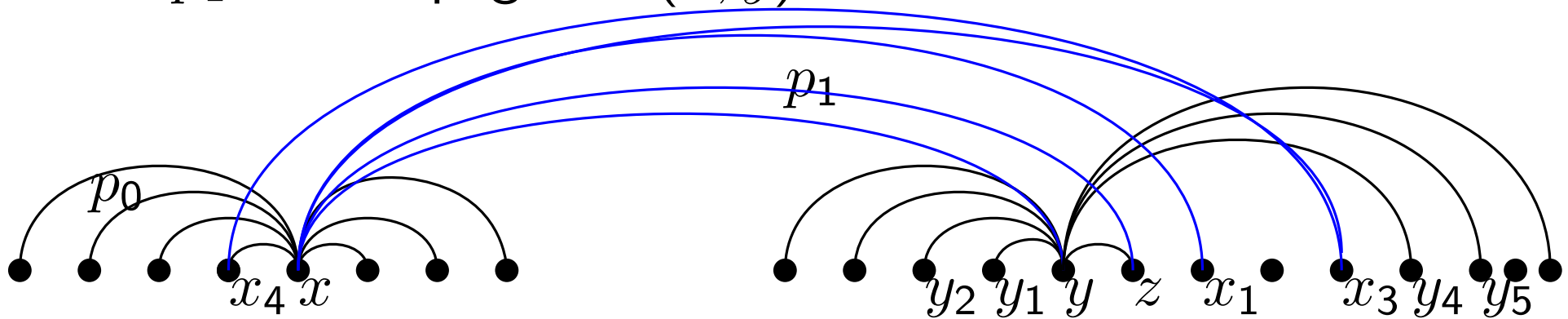
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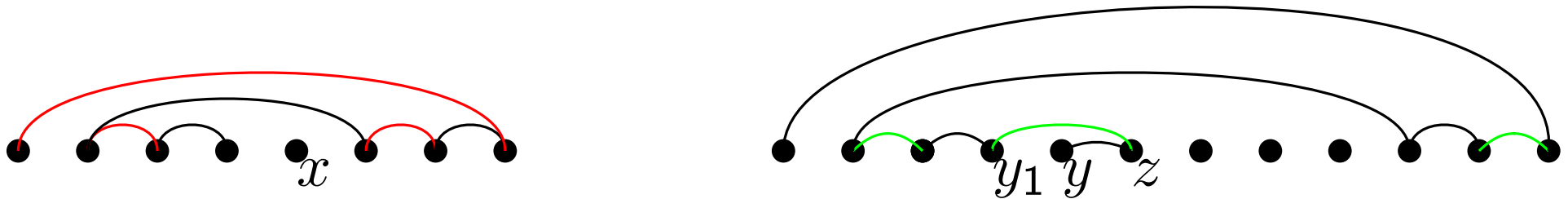
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# Results

**Theorem:** Directed monotone outerplanar graphs can be linearly embedded in 6 pages.

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can be improved to ...

**Theorem:** Directed monotone outerplanar graphs can be linearly embedded in 5 pages

# Our conjecture

Non-monotone outerplanar graphs should be doable in about 12-22 pages.

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Thank you !