From local pair-crossing number to local crossing number

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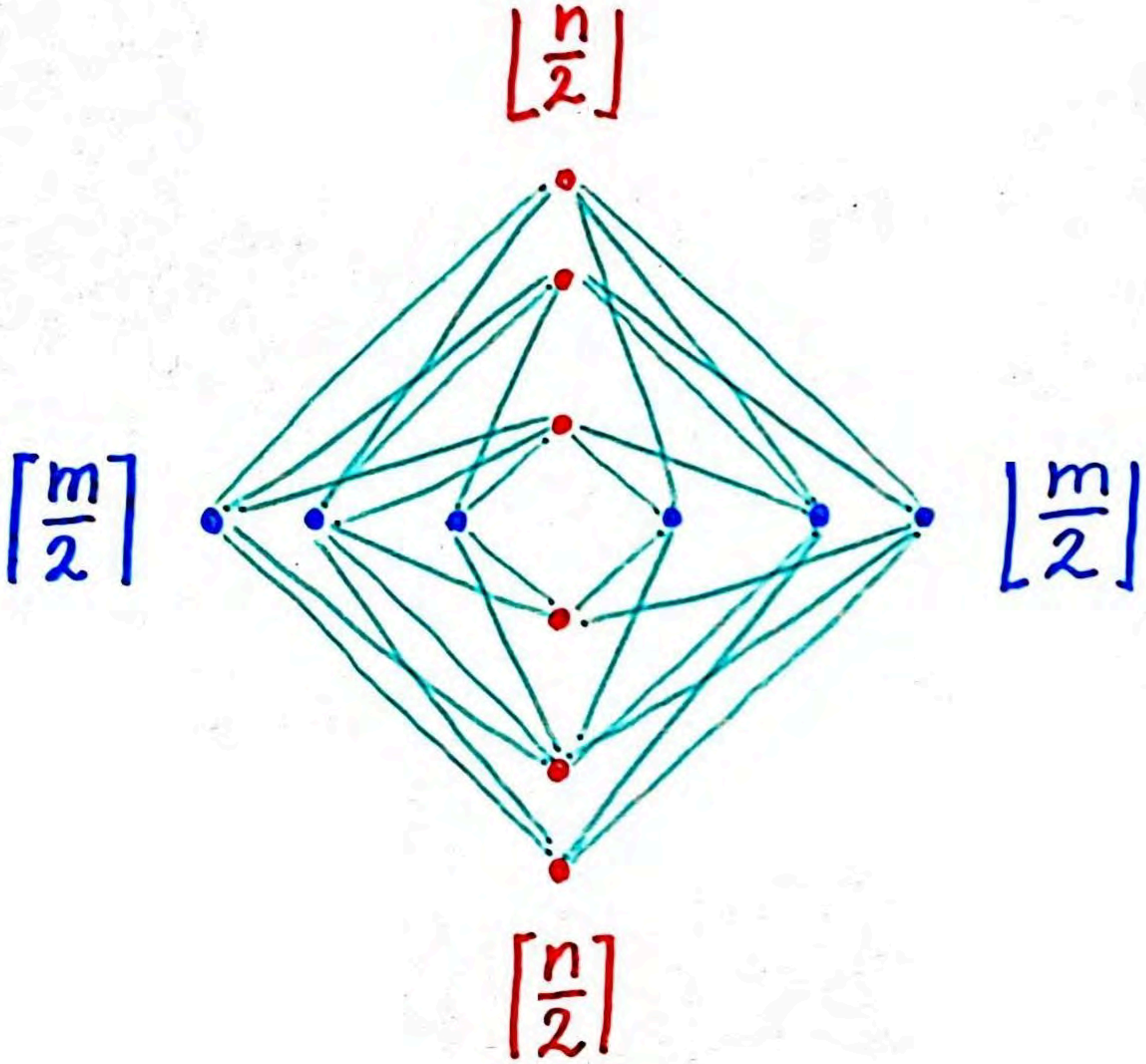
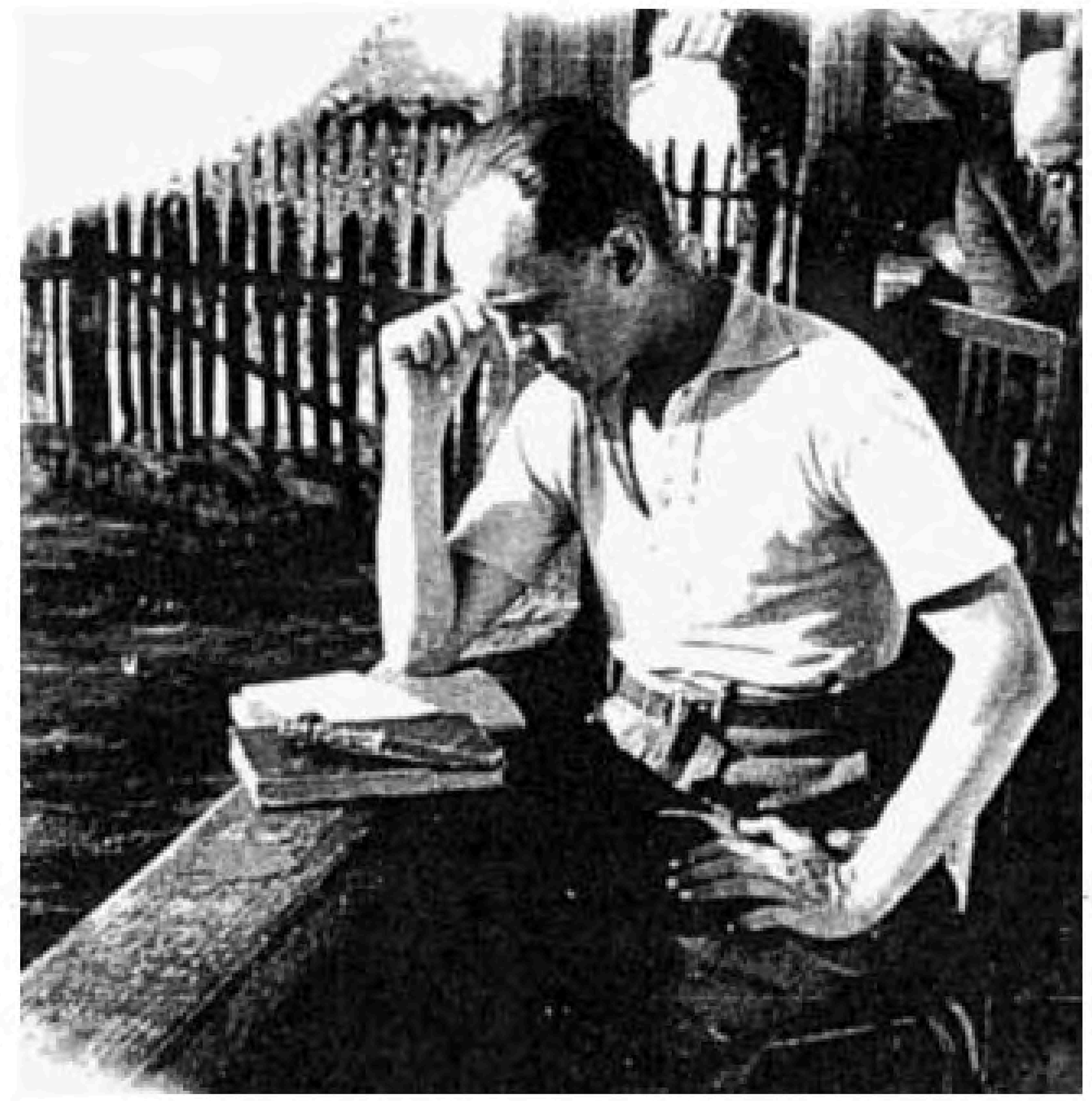
TURÁN'S BRICK FACTORY PROBLEM





Zarankiewicz 1954

$$cr(K_{m,n}) \leq \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

WHICH CROSSING NUMBER ?

crossing number of G $\quad\quad$ $\mathrm{cr}(G)$

minimum number of crossing points between the edges in the best curvilinear drawing of G

pair-crossing number of G $\quad\quad$ $\text{pair-cr}(G)$

minimum number of crossing pairs of edges in the best curvilinear drawing of G

Problem (P.-Tóth 00)

Does there exist a graph G, for which $\mathrm{cr}(G) \neq \text{pair-cr}(G)$?

$\mathrm{cr}(G) = O((\text{pair-cr}(G))^2)$ $\quad$ Matoušek 13

Tóth 13, Karl-Tóth 23

$\mathrm{cr}(G) = O((\text{pair-cr}(G))^{3/2})$ $\quad$ O. Solé Pi 25

Local Crossing Numbers

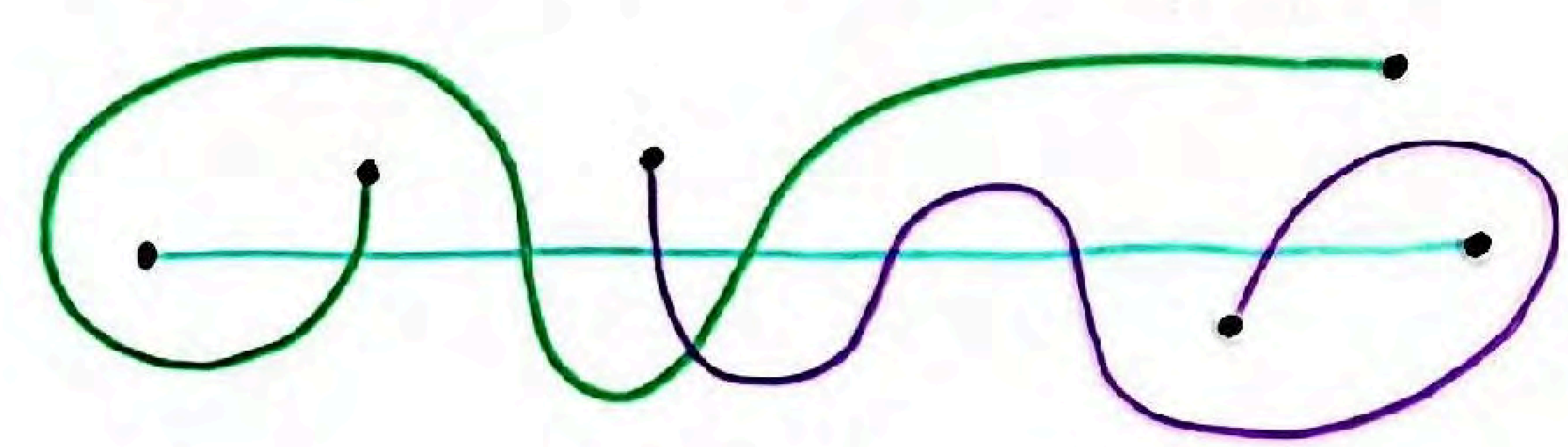
local crossing number of G $\qquad$ loc-cr(G)

the smallest upper bound on the number of crossings along an edge in the best drawing of G

local pair-crossing number of G $\qquad$ loc-pair-cr(G)

the smallest upper bound on the number of edges that cross a single edge in the best drawing of G (without multiplicities)

$$\text{loc-pair-cr}(G) \leq \text{loc-cr}(G)$$



$2 < 7$

LOCAL CROSSING NUMBERS

local crossing number of G loc-cr(G)

the smallest upper bound on the number of crossings along an edge in the best drawing of G

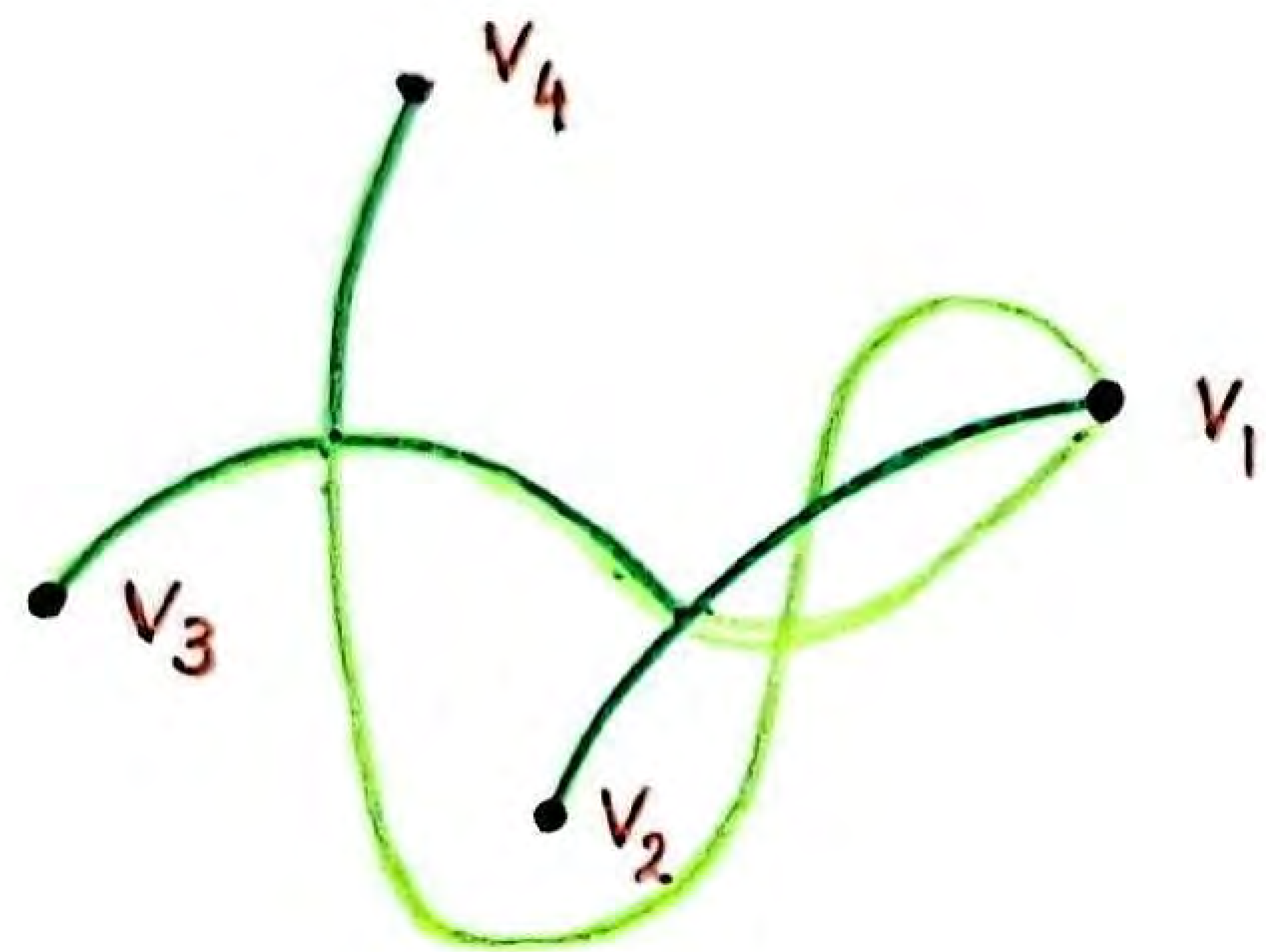
local pair-crossing number of G loc-pair-cr(G)

the smallest upper bound on the number of edges that cross a single edge in the best drawing of G (without multiplicities)

Theorem (Schaefer-Štefankovič 04)
Let G be a graph with loc-pair-cr$(G) \leq k$.
Then we have loc-cr$(G) \leq 2^k$

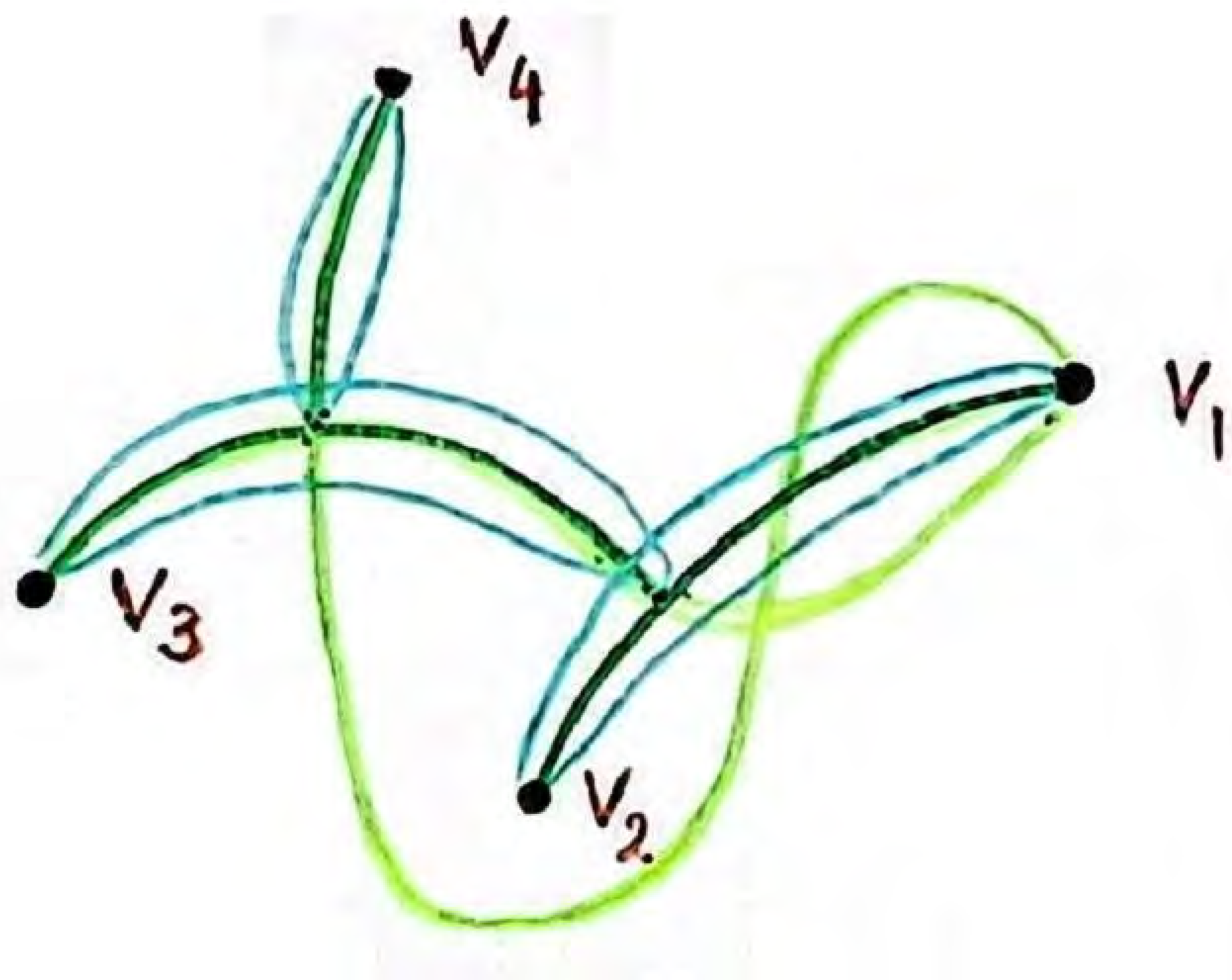
Theorem (Fox-P.-Suk 25)
Let G be a graph with loc-pair-cr$(G) \leq k$.
Then we have loc-cr$(G) \leq k^3 + 2k^2 + k$

Proof idea: Splitting the plane into two



non-selfintersecting spanning tree

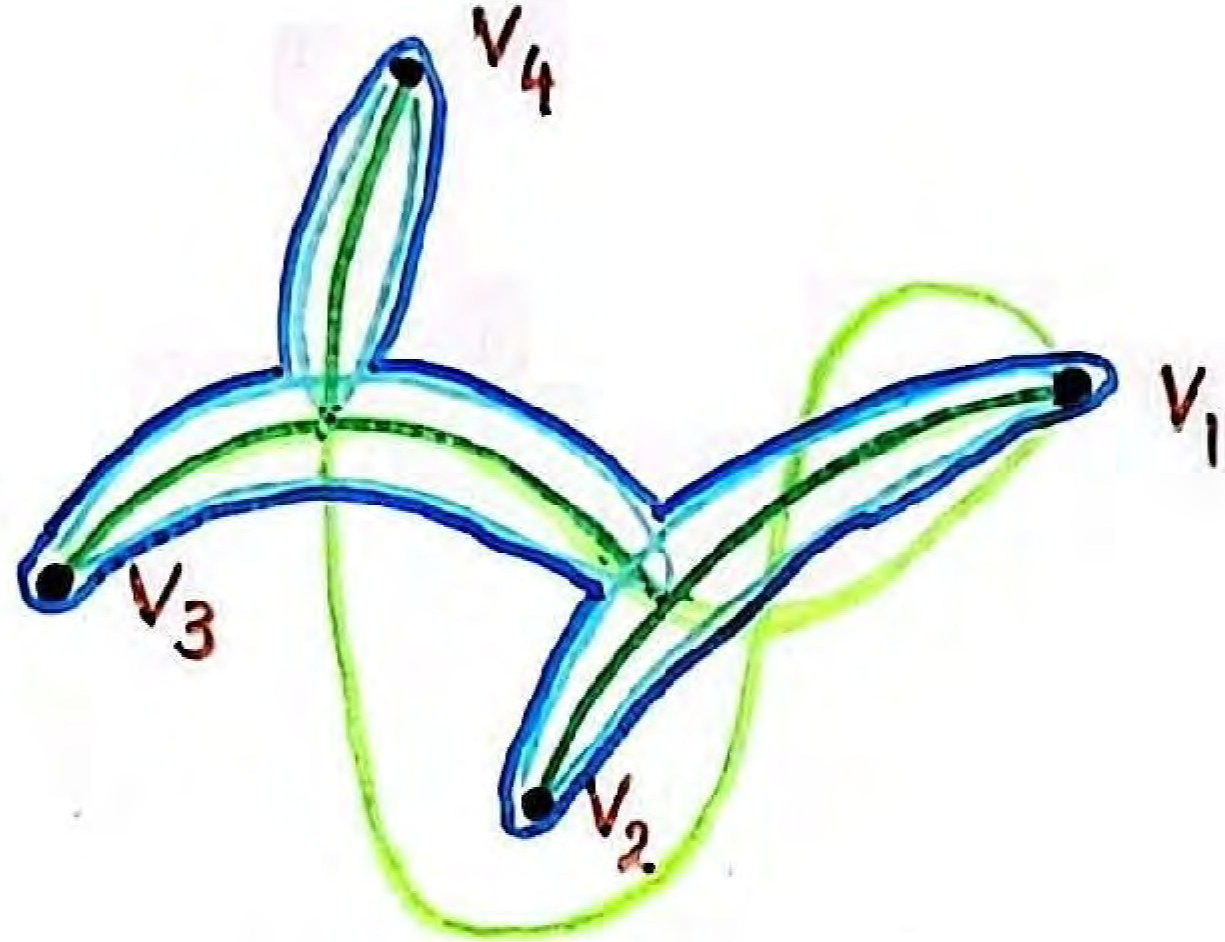
Proof idea: Splitting the plane into two



non-selfintersecting spanning tree

cactus structeure - small neighborhood

Proof idea: Splitting the plane into two

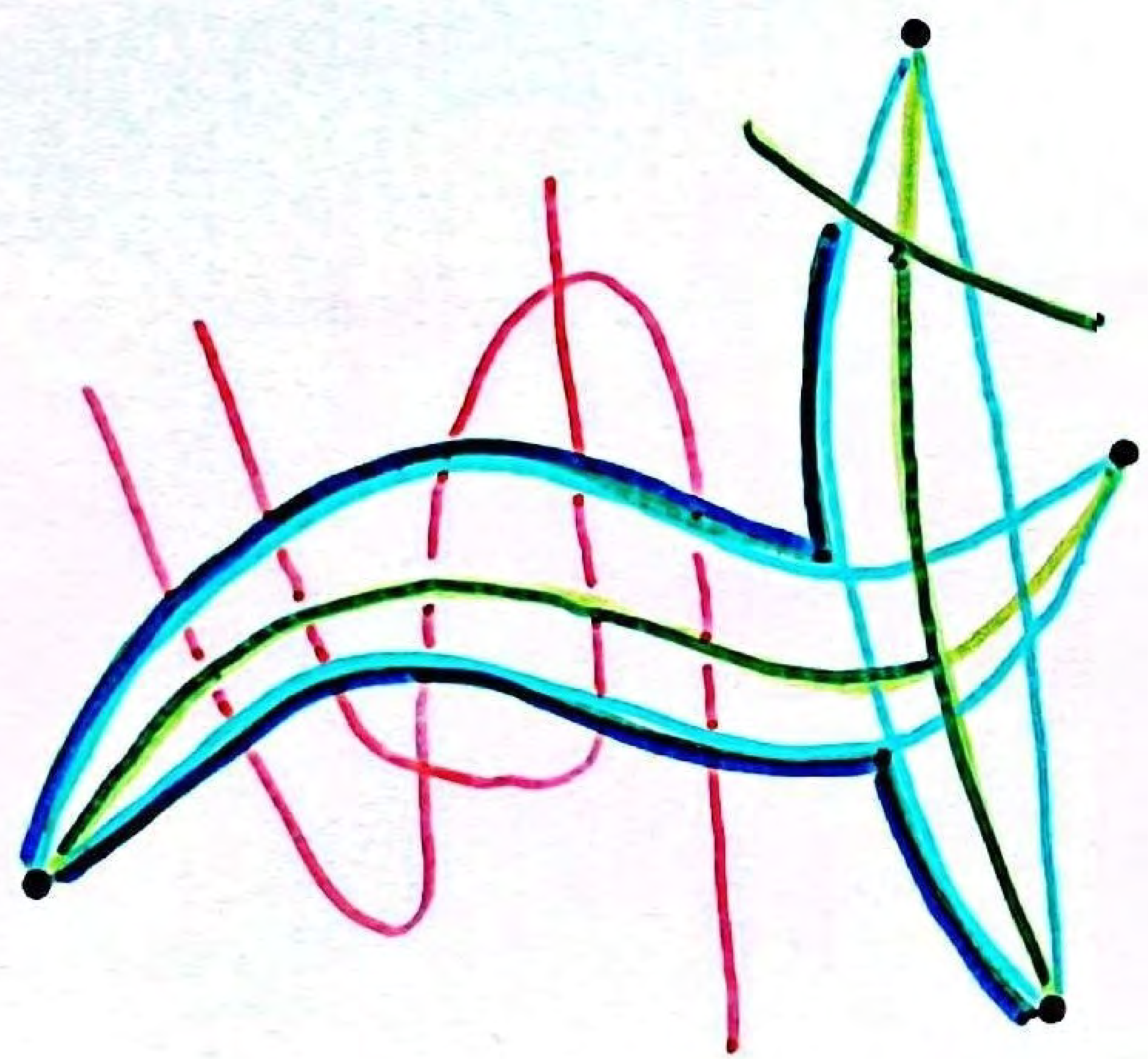


non-selfintersecting spanning tree

cactus structure - small neighborhood

enclosing in a capsule

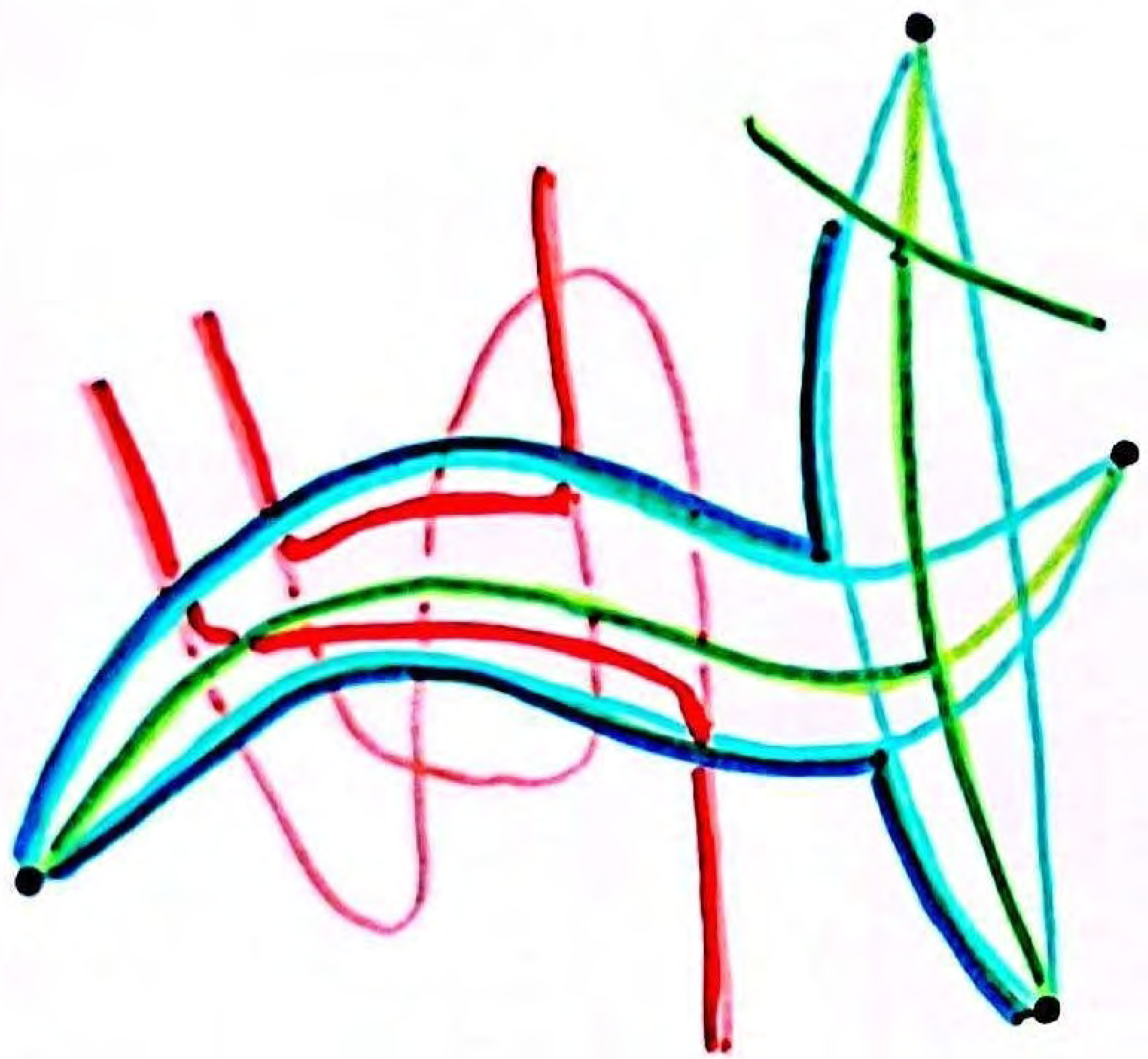
REDRAWING INSIDE THE CAPSULE



$\forall e \in E(G)$ has $\leq k$ arcs inside the capsule

$\leq k+1$ arcs outside the capsule

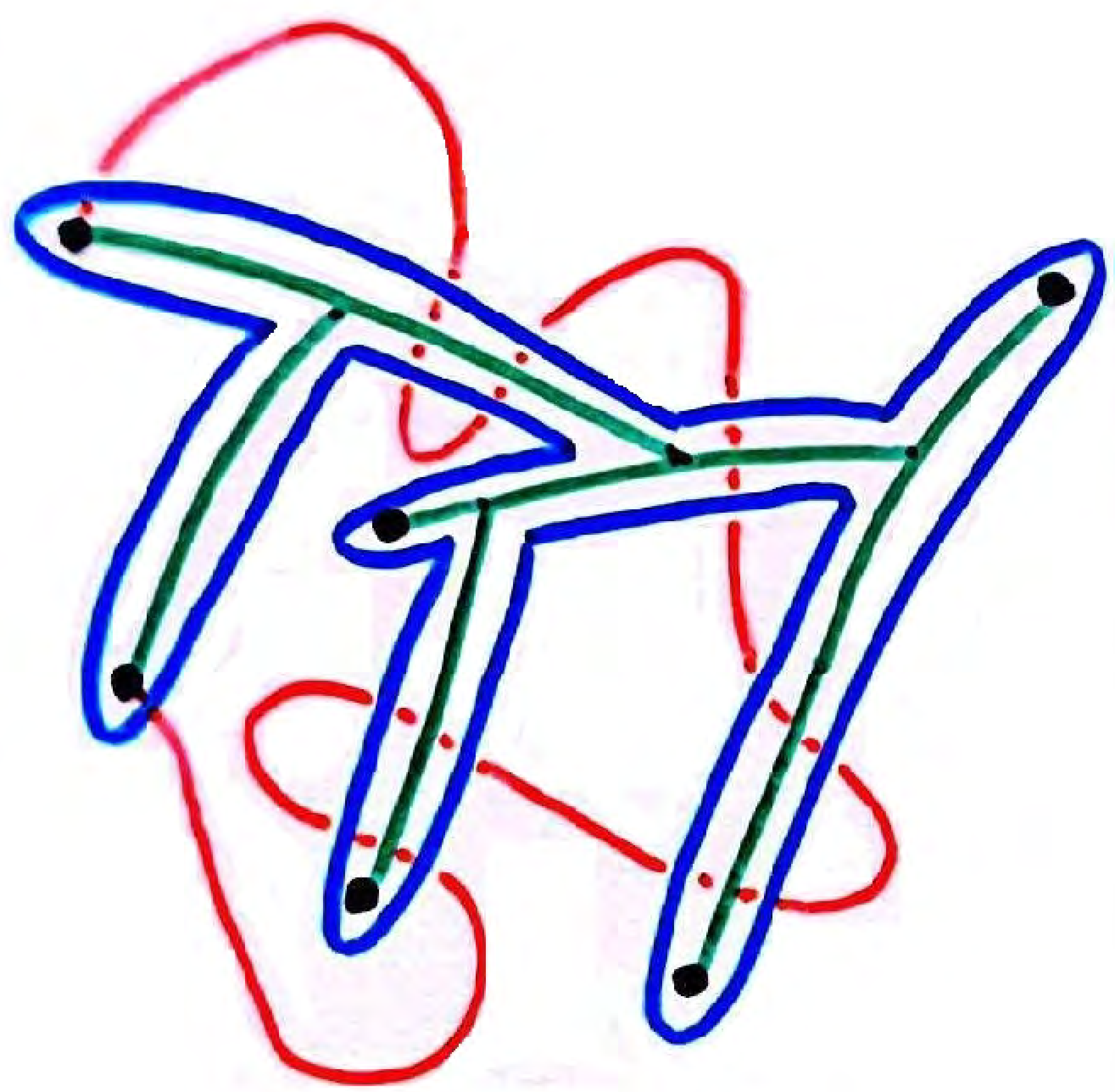
REDRAWING INSIDE THE CAPSULE



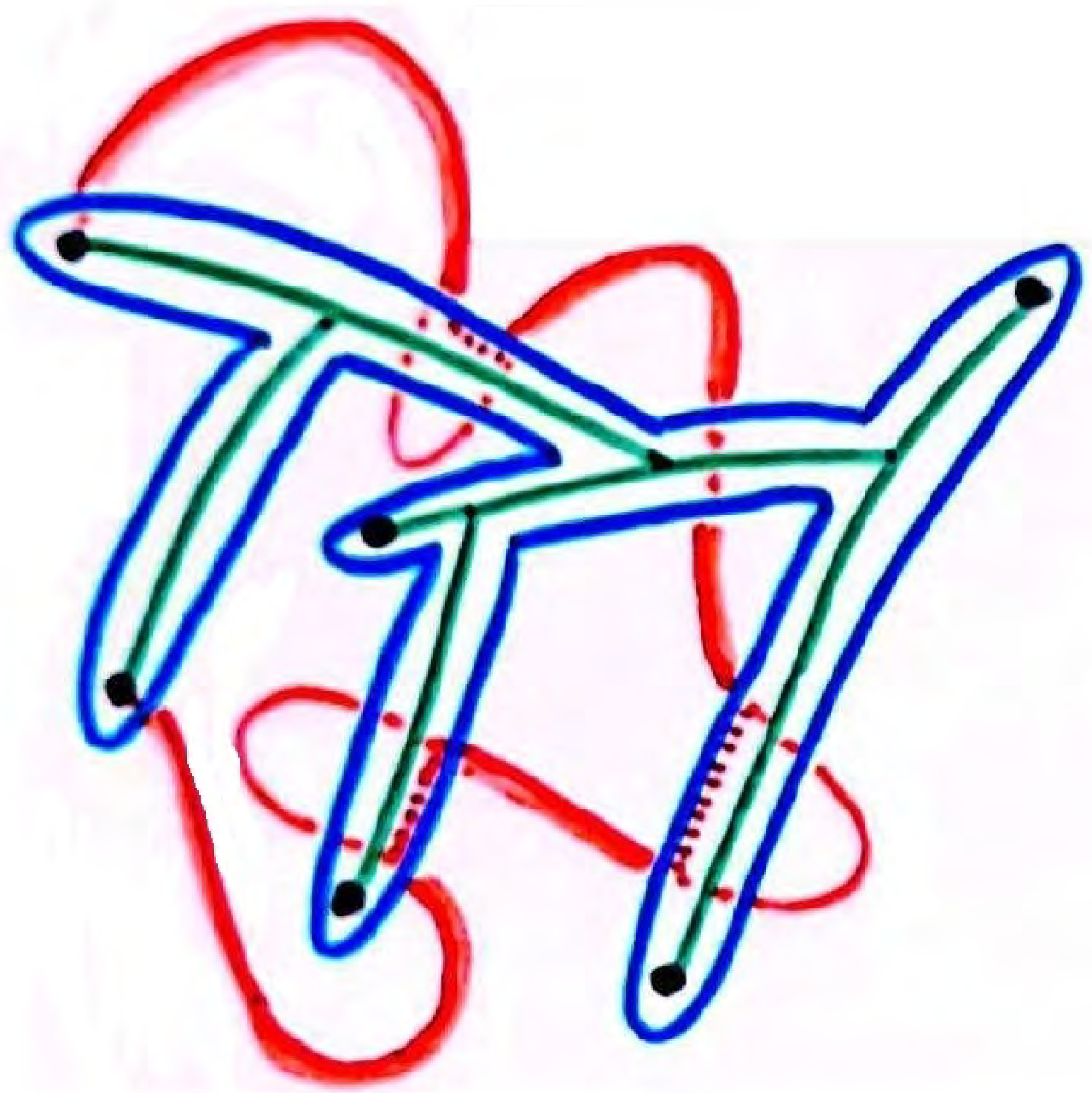
$\forall e \in E(G)$ has $\leq k$ arcs inside the capsule
$\leq k+1$ arcs outside the capsule

$\forall$ internal arc has ≤ 1 crossing with ≤ 1 arc of each of the $\leq k$ edges that intersect the given part of the capsule

REDRAWING OUTSIDE THE CAPSULE

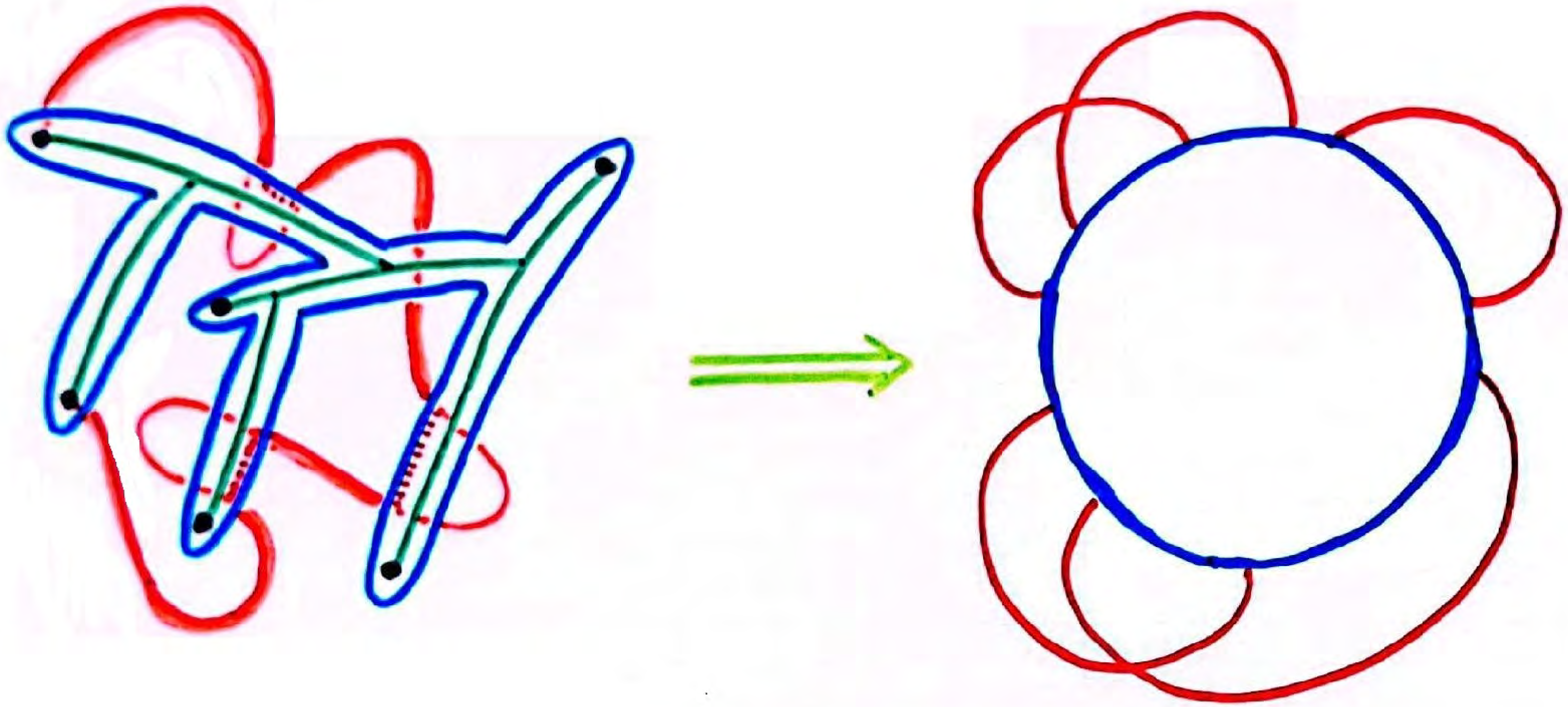


REDRAWING OUTSIDE THE CAPSULE



$\forall 2$ redrawn external arcs cross in ≤ 1 point

REDRAWING OUTSIDE THE CAPSULE



no "unnecessary" crossings

$\forall 2$ redrawn external arcs cross in ≤ 1 point

$\forall 2$ edges have $\leq (k+1)^2$ external crossings

$\Longrightarrow$ $\forall$ edge has $\leq k(k+1)^2$ external crossings

- During redrawing, the cyclic order of the edges incident to any vertex remains unchanged ("rotation system")

- Theorems hold for multigraphs

- Theorem (Fox - P. - Suk 2025+)
  Let G be a graph with $\text{loc-pair-cr}(G) \leq k$.
  Then we have $\text{loc-pair-cr}(G) = O(k^{2.5})$.

- Corollary (Fox - P. - Suk)
  For any graph G, we have $\text{cr}(G) = O\left((\text{pair-cr}(G))^{5/3}\right)$