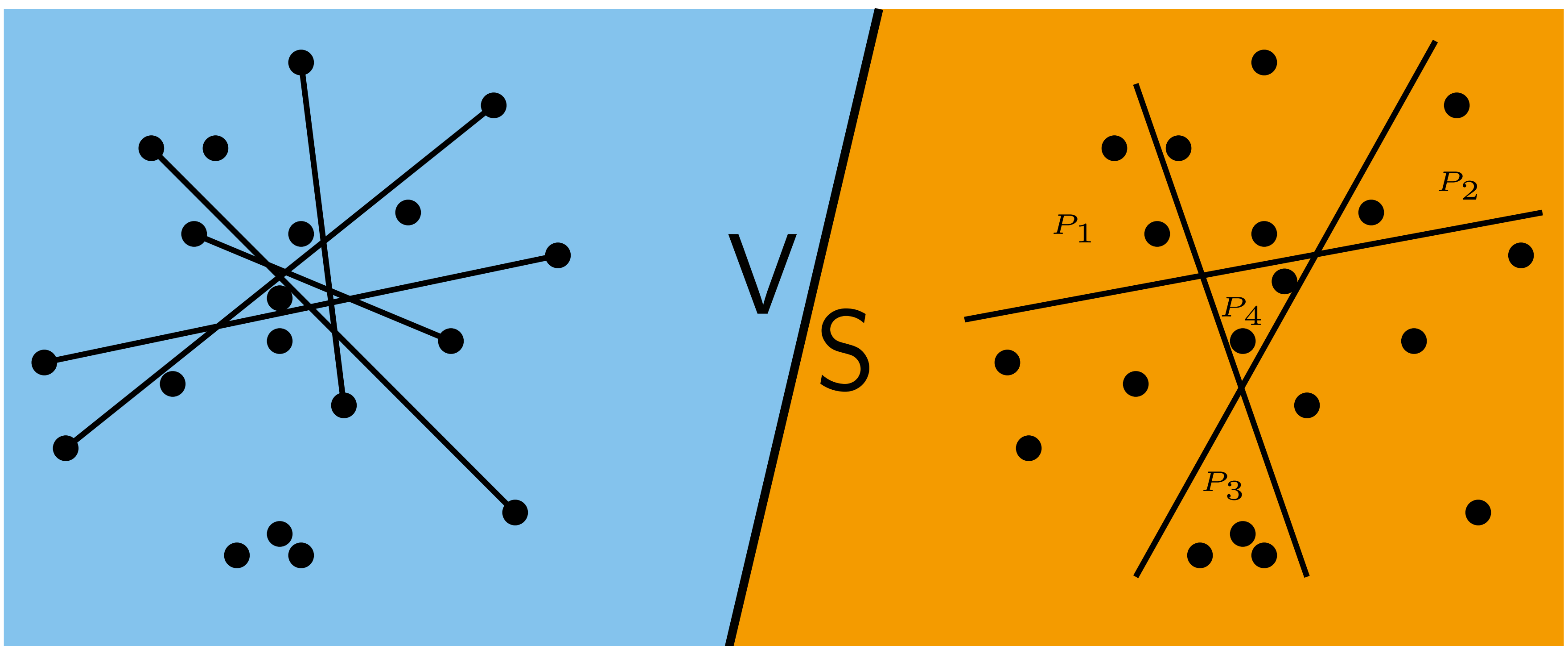
Crossing and non-crossing families

Todor Antić, Martin Balko and Birgit Vogtenhuber

GD 2025



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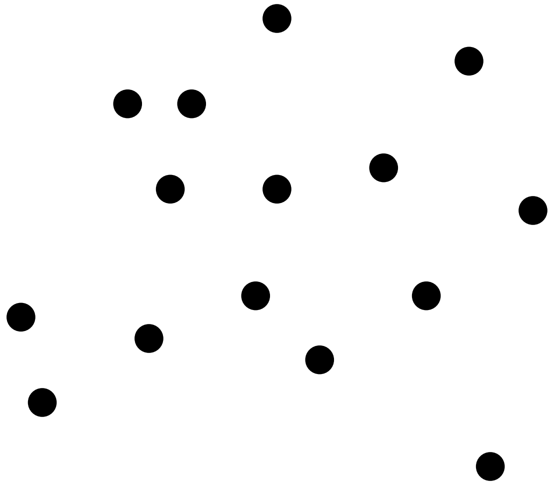

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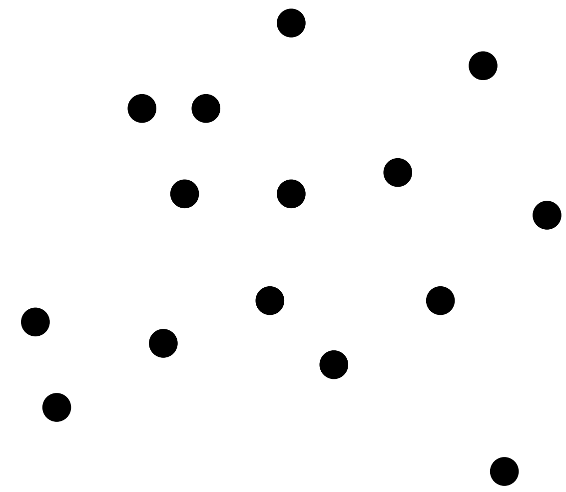
Crossing family in $P \rightarrow$ set of pairwise crossing segments between points in P



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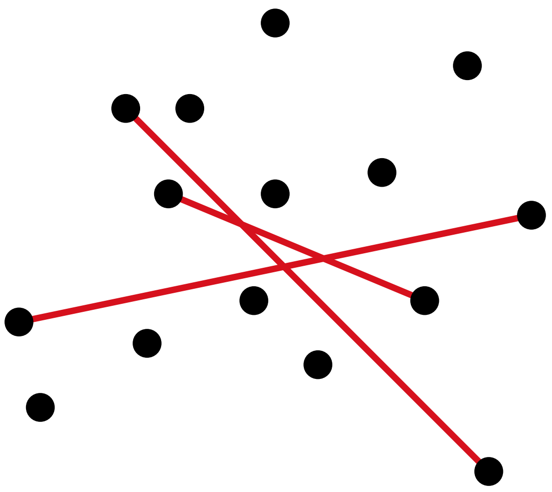

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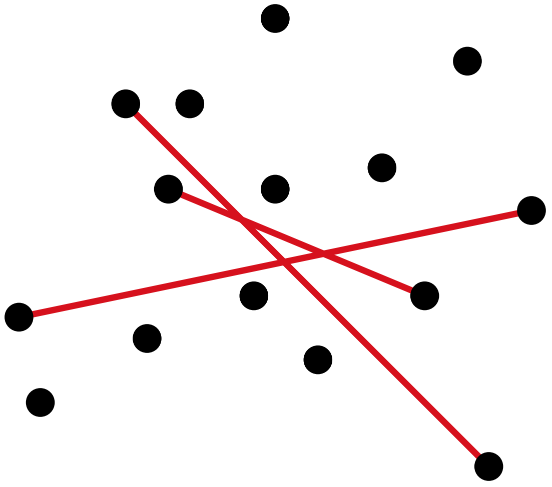

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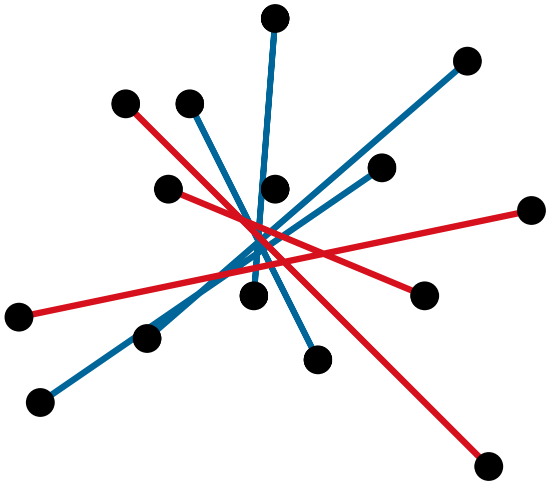

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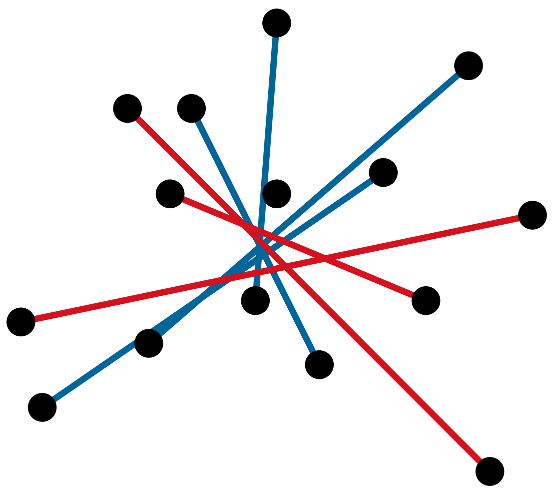

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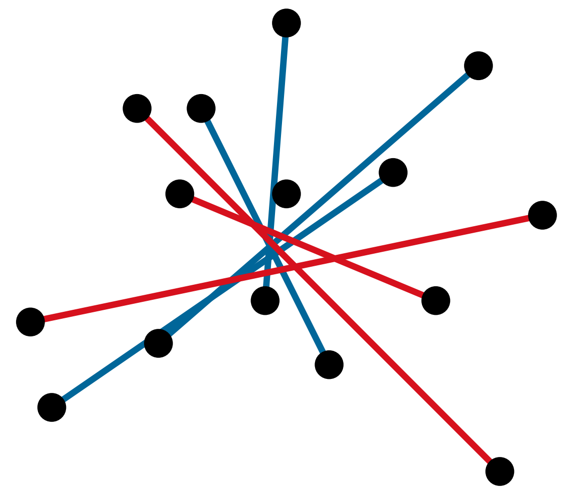

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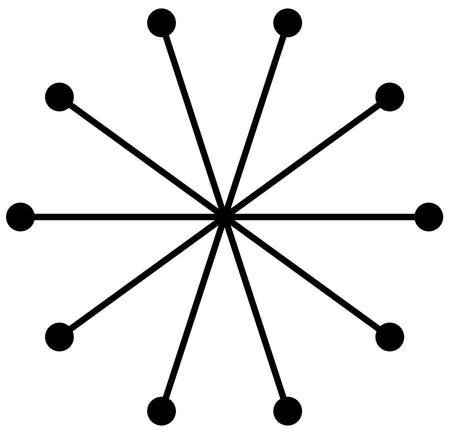

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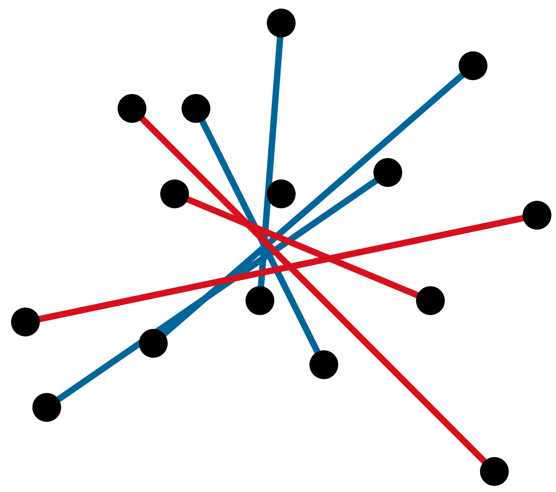

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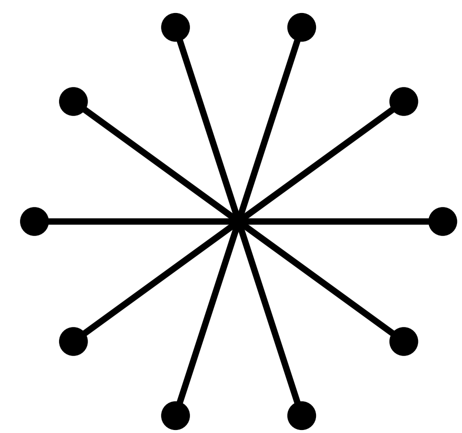

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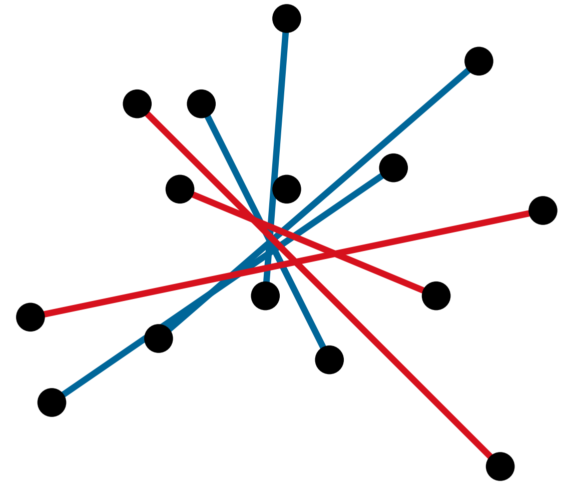

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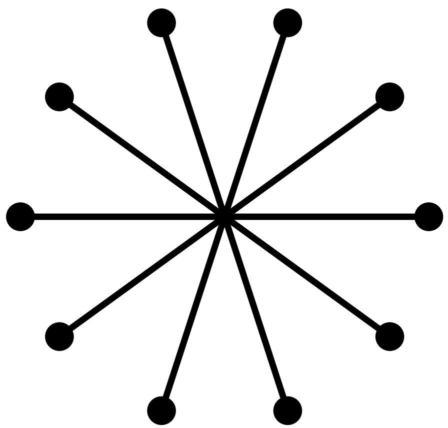

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Big question: Is $f(n)$ linear?

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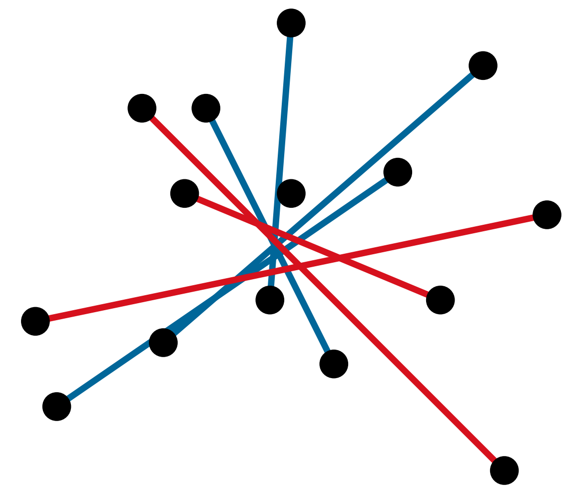

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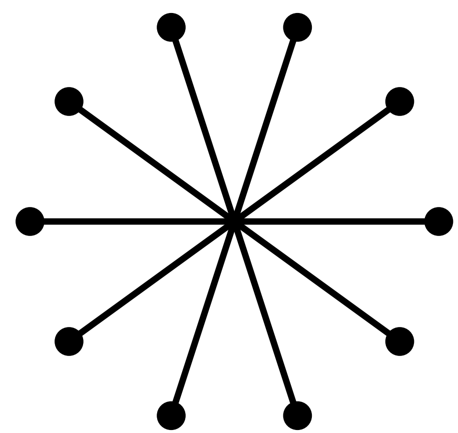

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Big question: Is $f(n)$ linear?

Known:

- $f(n) \geq \frac{n}{2^{O(\sqrt{\log n})}}$ [Pach, Rubin, Tardos]
- $f(n) \leq \lceil \frac{8n}{41} \rceil$ [Aichholzer et al.]

Non-crossing families

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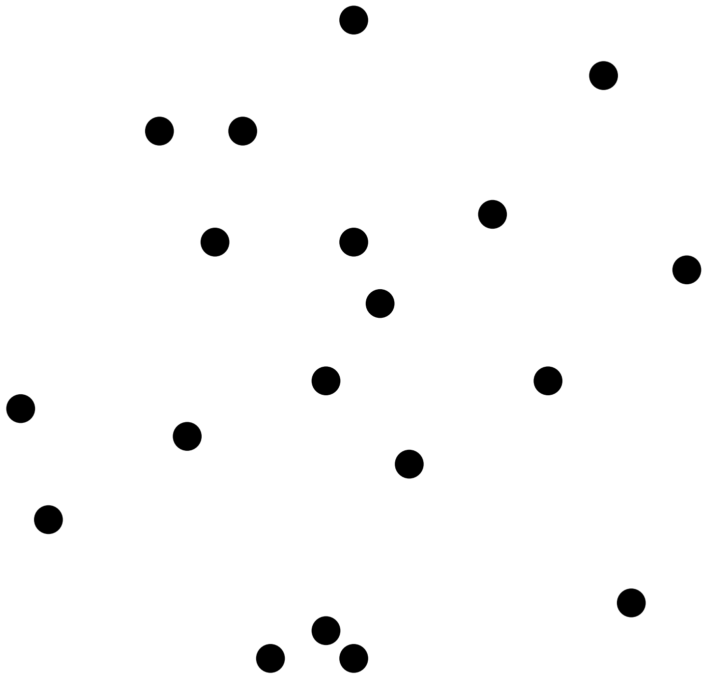

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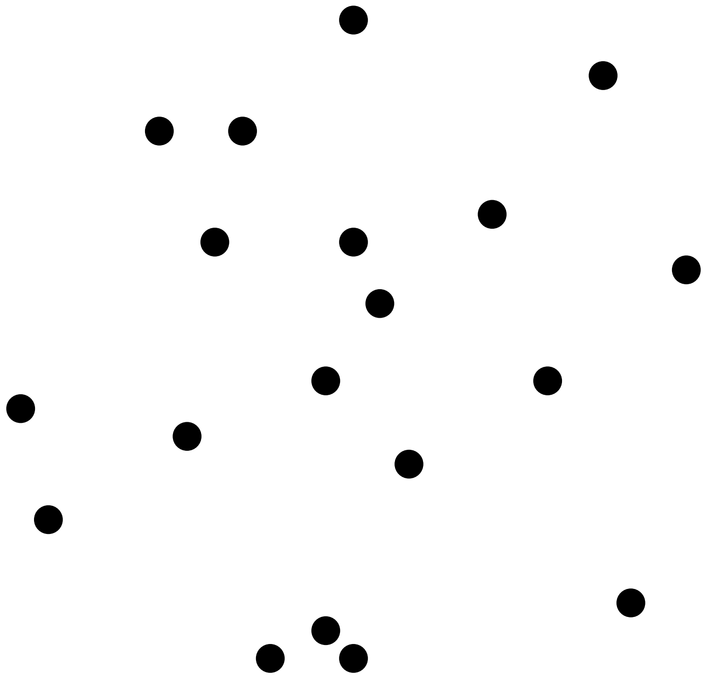
Non-crossing family of size k in P:



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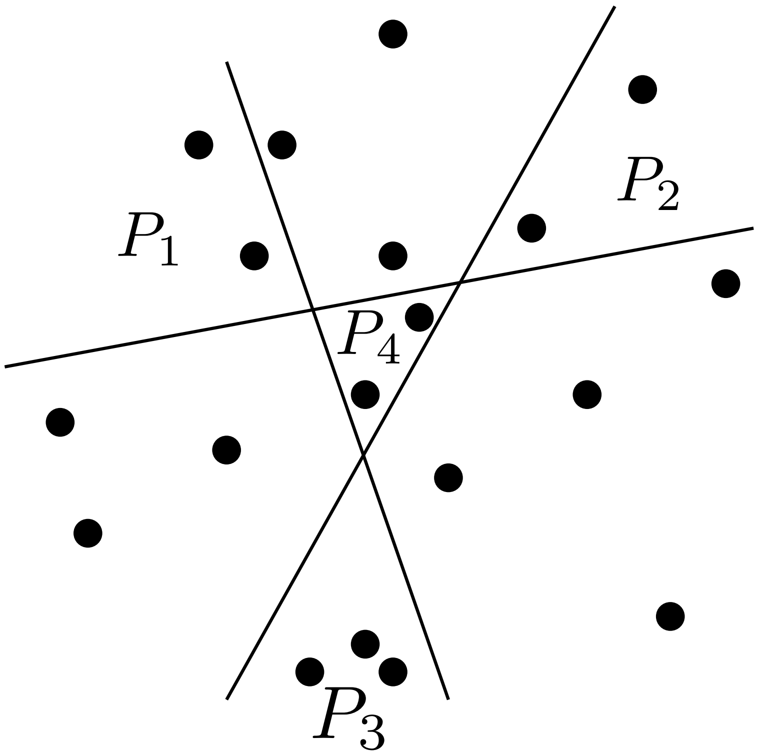

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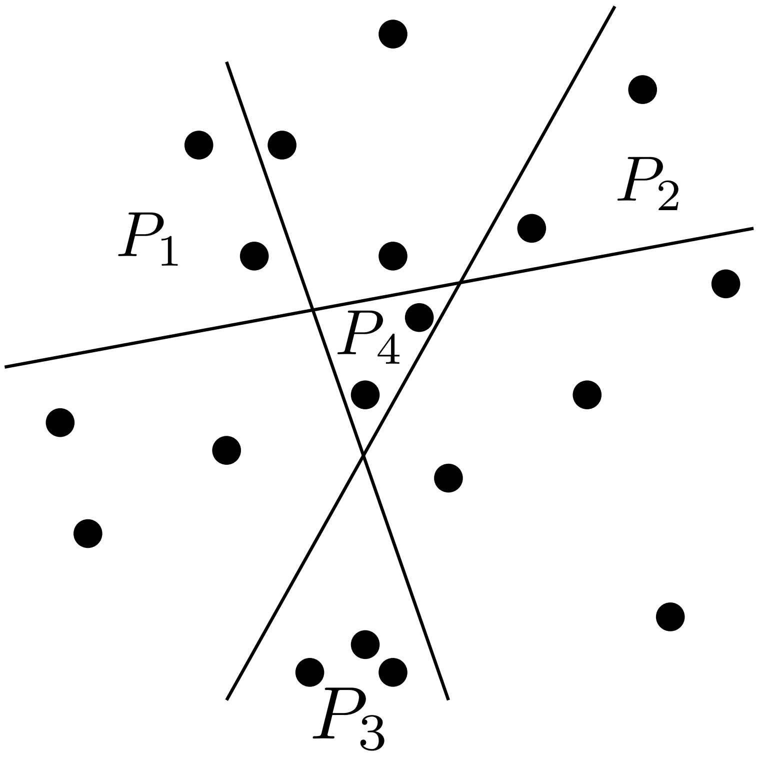

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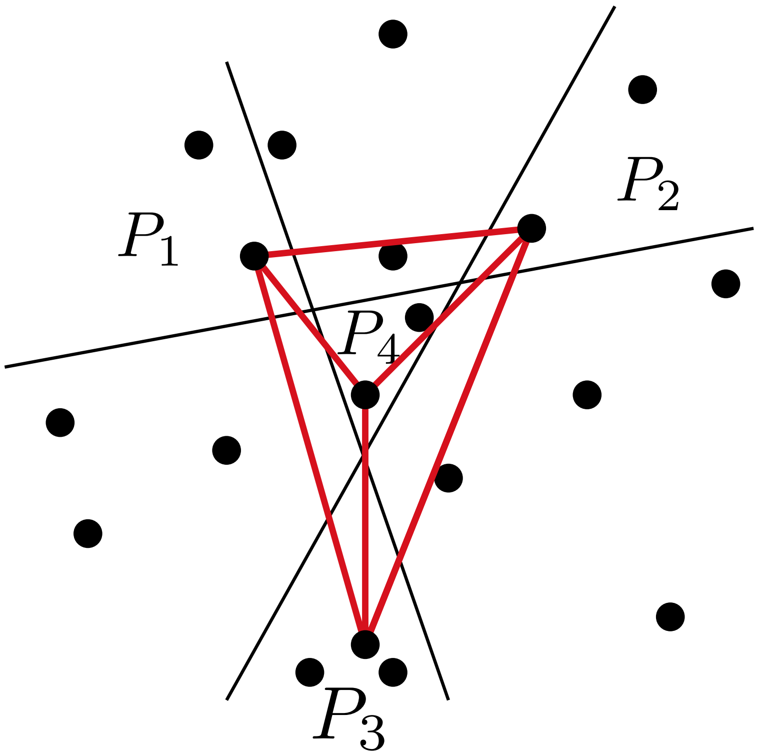

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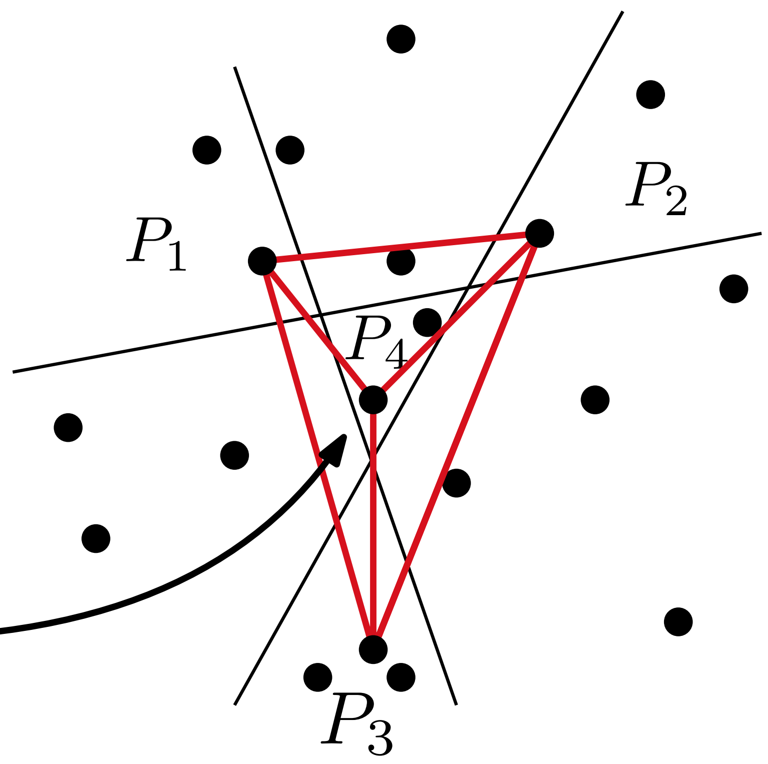

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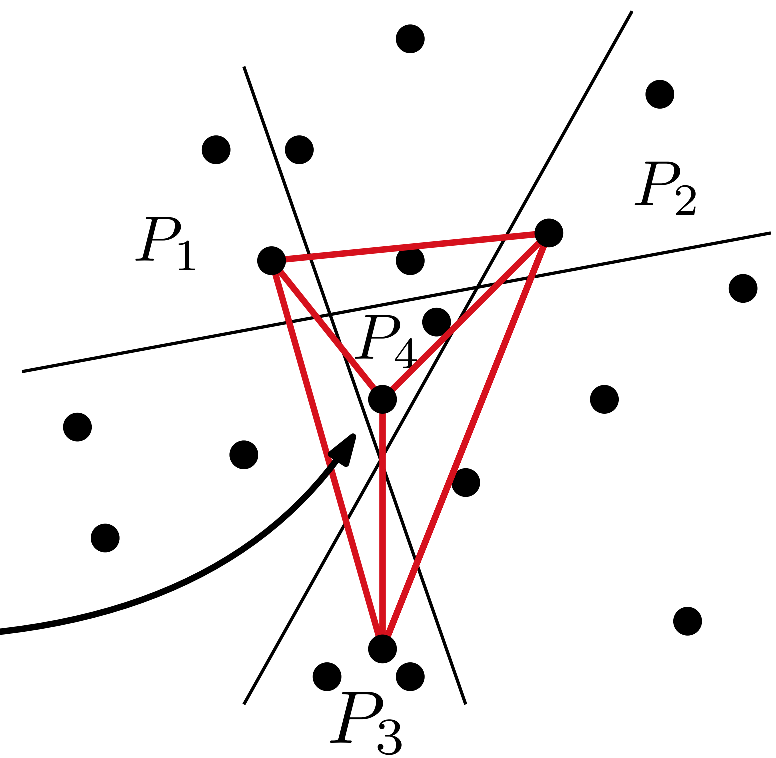

Question:
If P does not have a non-crossing family of size m, how large crossing family can we find in P?

Edge-partitioning of complete geometric graphs

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Complete geometric graph $\rightarrow$ drawing of K_n with edges drawn as line segments between vertices

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Known:

- If the vertex set has a linear-sized crossing family then such c exists. [2006, BHR-CW]
- If the vertex set has a linear-sized non-crossing family then such c exists. [GD 2023, Pach, Schnider, Saghafian]

First result

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Collection of sets $C_1, C_2, \ldots, C_k$ is a **convex bundle** if for each choice of $c_i \in C_i$, the set $c_1, c_2, \ldots, c_k$ is in convex position.

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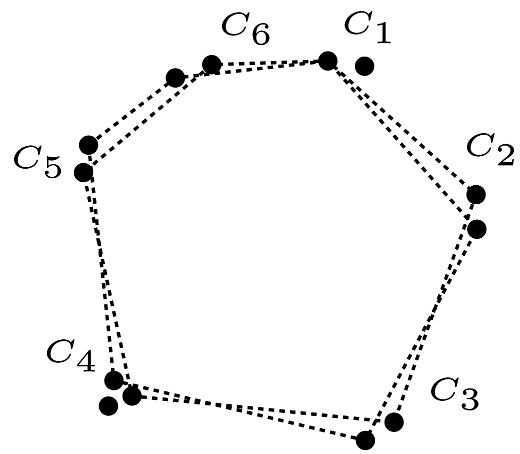

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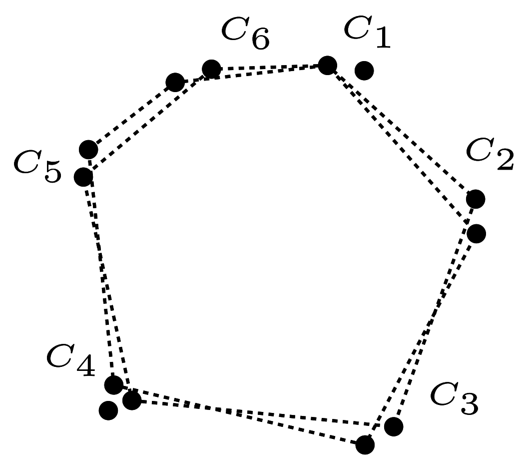

Thm: There is a constant $C > 0$ such that, for all positive integers m and k, every set of at least Ckm points in the plane in general position contains either a convex bundle of size k and width m or a non-crossing family of size m.

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Proof: Modification a result of Pach, Rubin and Tardos

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Thm: There is a constant $C' > 0$ such that, for every positive integer m, every set of $n \geq C'm$ points in the plane in general position contains either a non-crossing family of size m or a crossing family of size $n/2^{O(\sqrt{\log m})}$.

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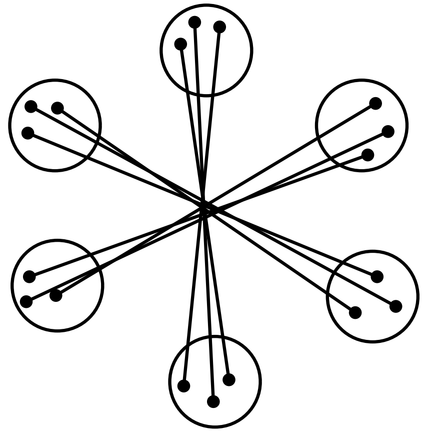

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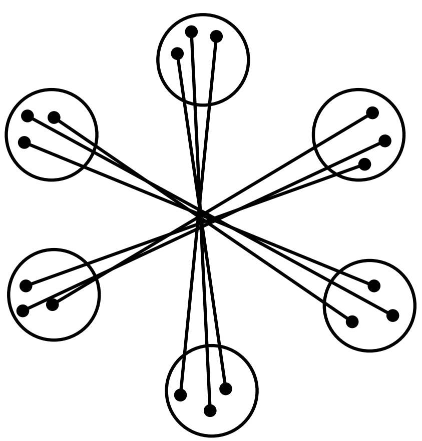
Proof:

- If a set has a non-crossing family of size m, we're done
- If not, we use first result to get a convex bundle of size k and width m (for suitable k)
- Pair the antipodes and use the second result
- If $k = \lfloor \frac{n}{Cm} \rfloor$ where C is constant from the first result, we're done



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Order type of a point set $P \rightarrow$ orientations of all point triples

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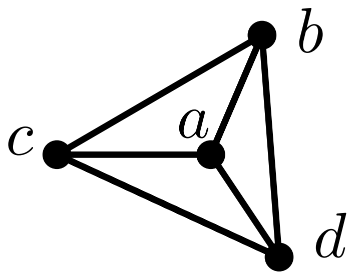

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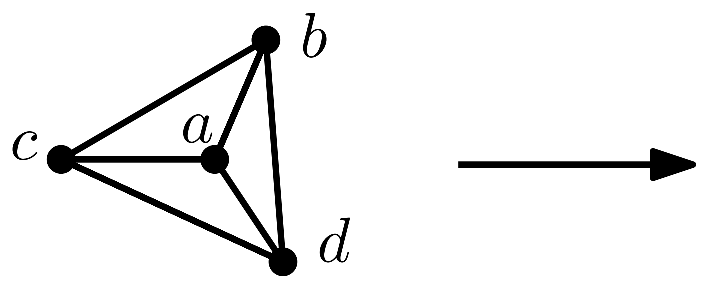

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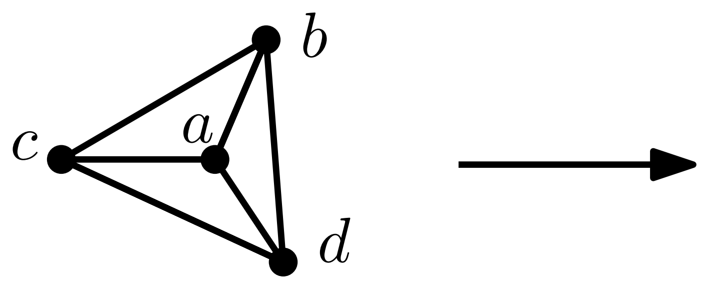

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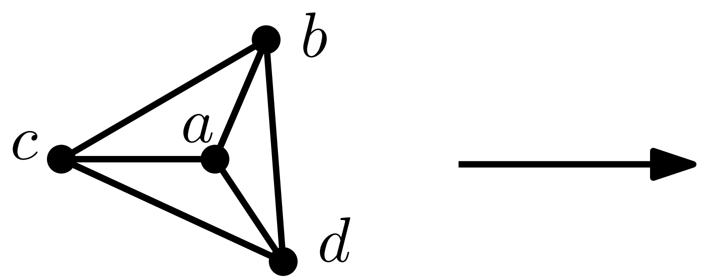

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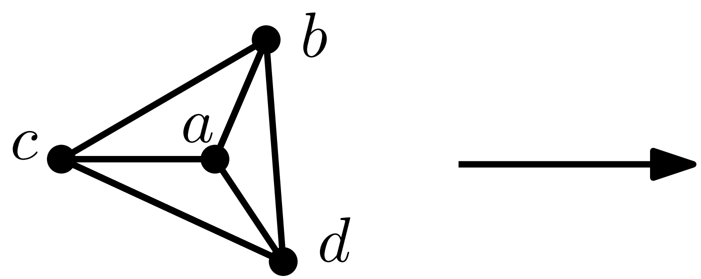

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$Y_1, \ldots, Y_r$ have the same-type property if any 3 of them do.

Thm:[Bárány,Valtr] There is a constant $c(r)$ such that for any pairwise disjoint point sets $X_1, X_2, \ldots, X_r$ in $\mathbb{R}^2$ whose union is in general position, there are point sets $Y_i \subseteq X_i$, with $|Y_i| \geq c(r)|X_i|$ having the same-type property.

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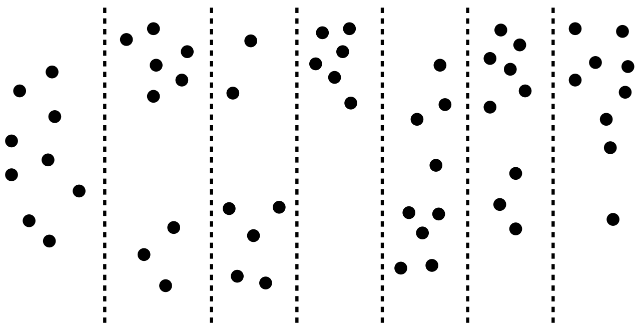

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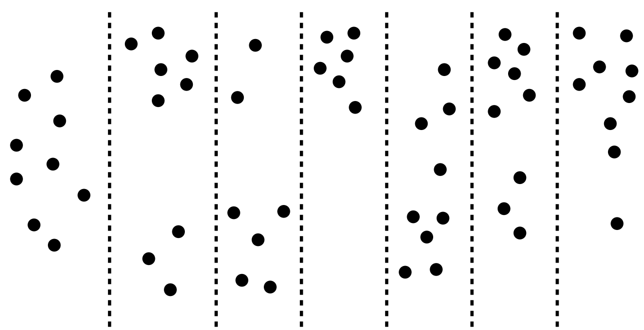
- Apply same-type lemma



First result - again

Thm: There is a constant $C > 0$ such that, for all positive integers m and k, every set of at least Ckm points in the plane in general position contains either a convex bundle of size k and width m or a non-crossing family of size m.

Proof: First step - preprocessing

- Let P be a set of Ckm points where $C = c(7)^{-1}c(5)^{-5}$
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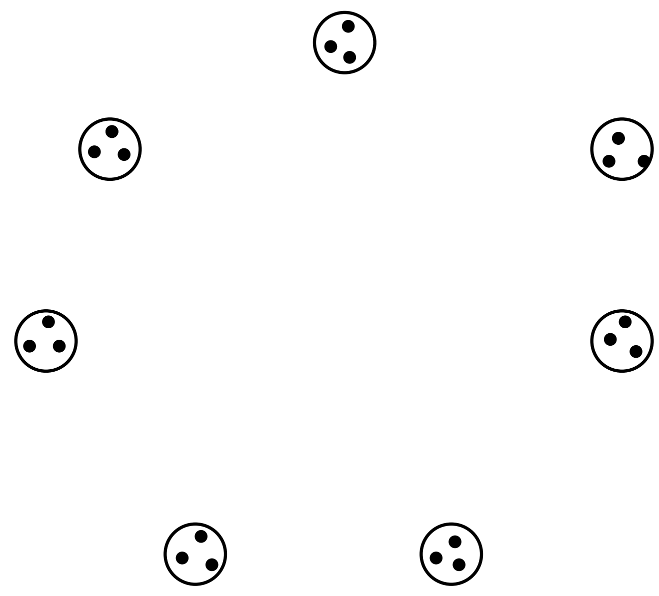

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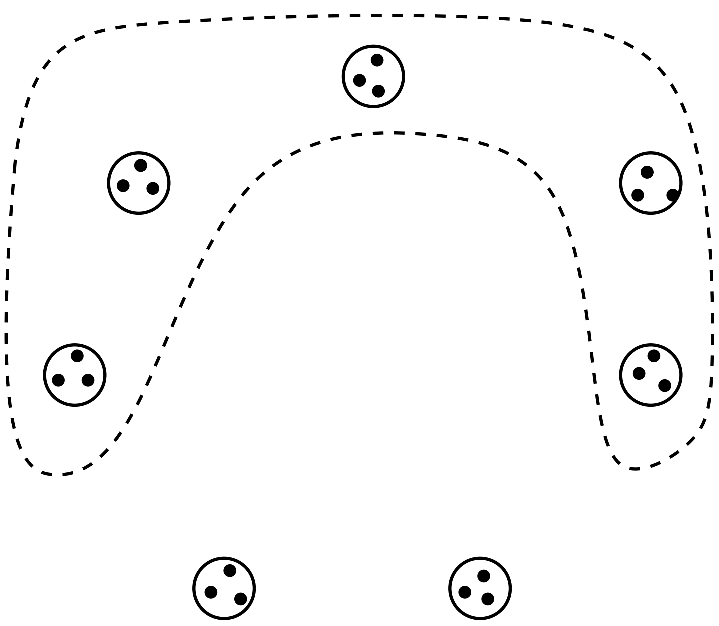

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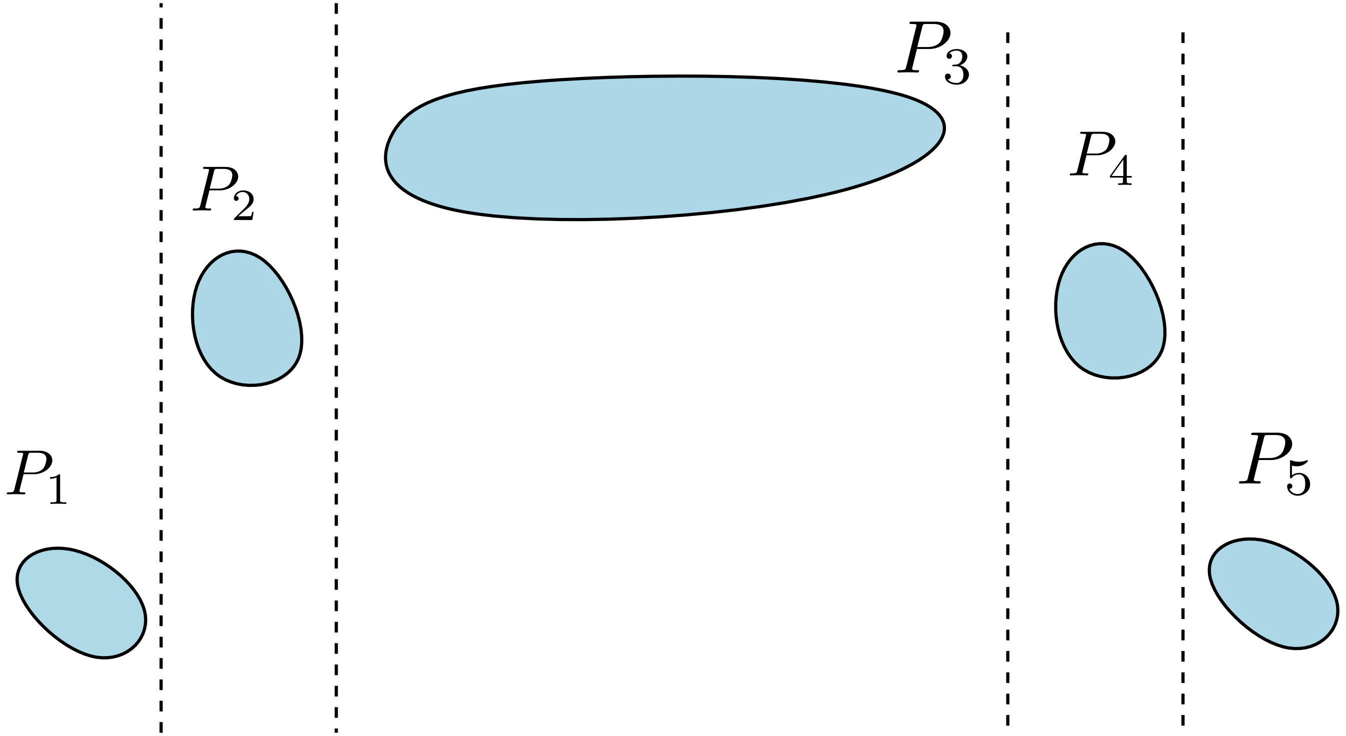

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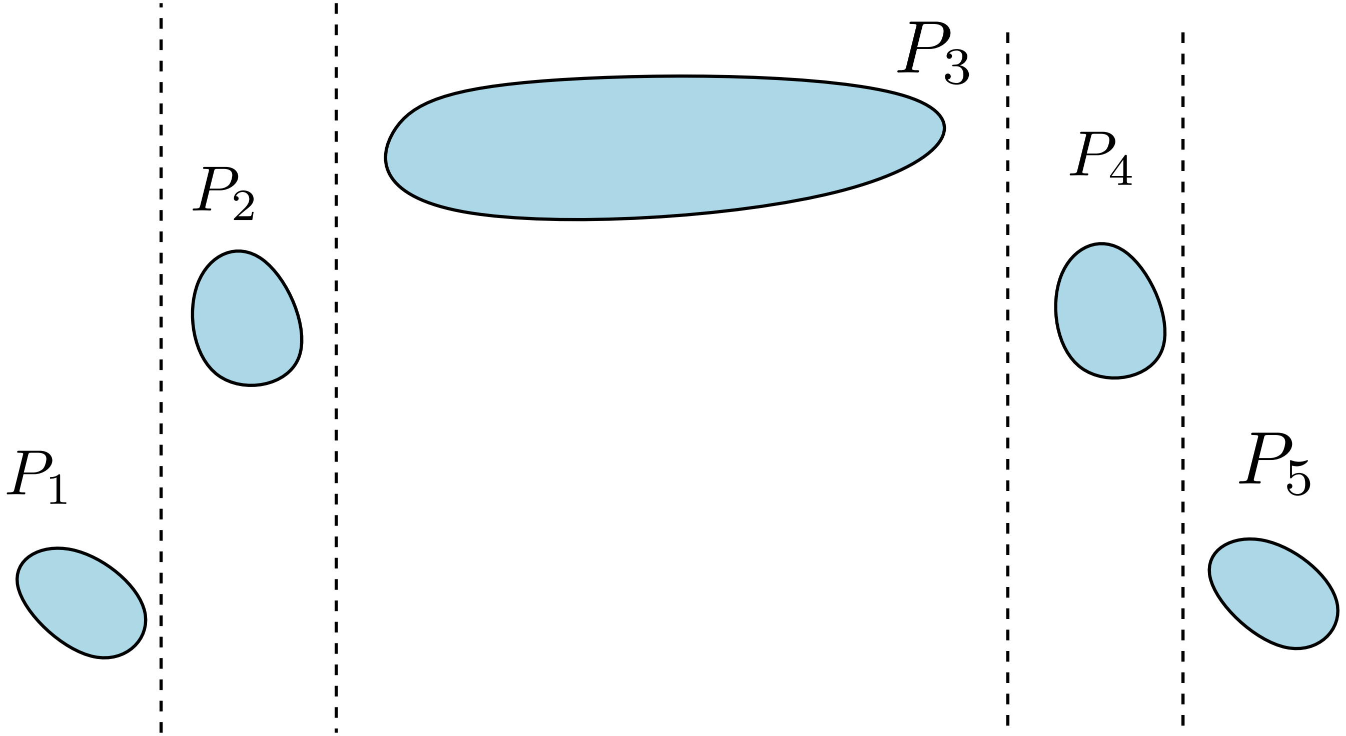

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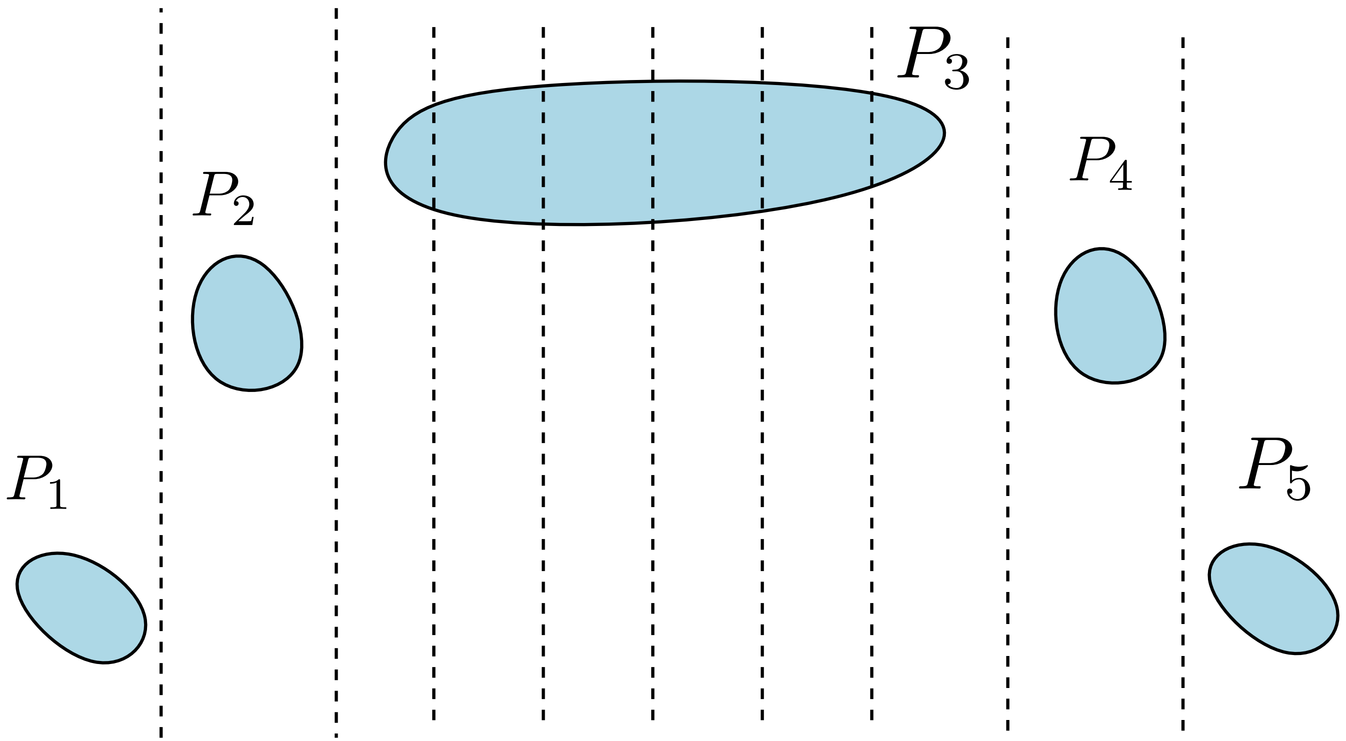

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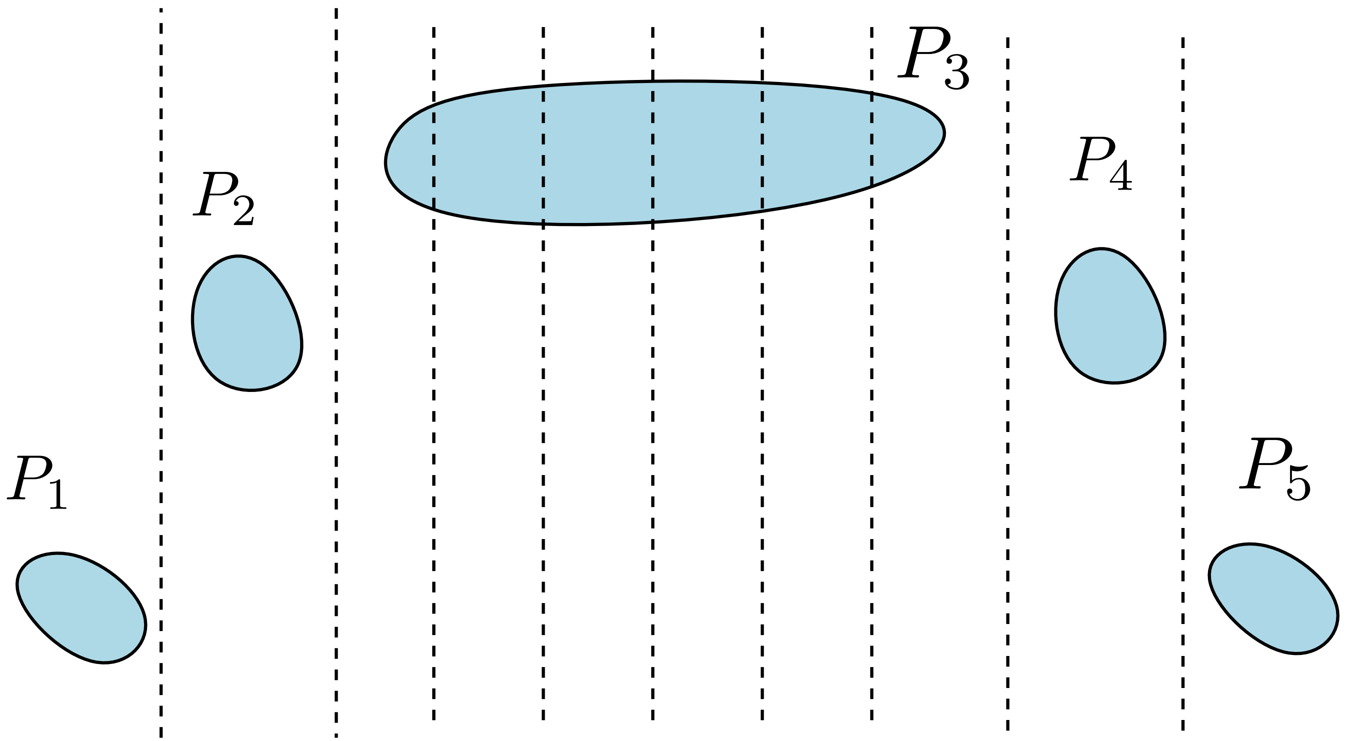

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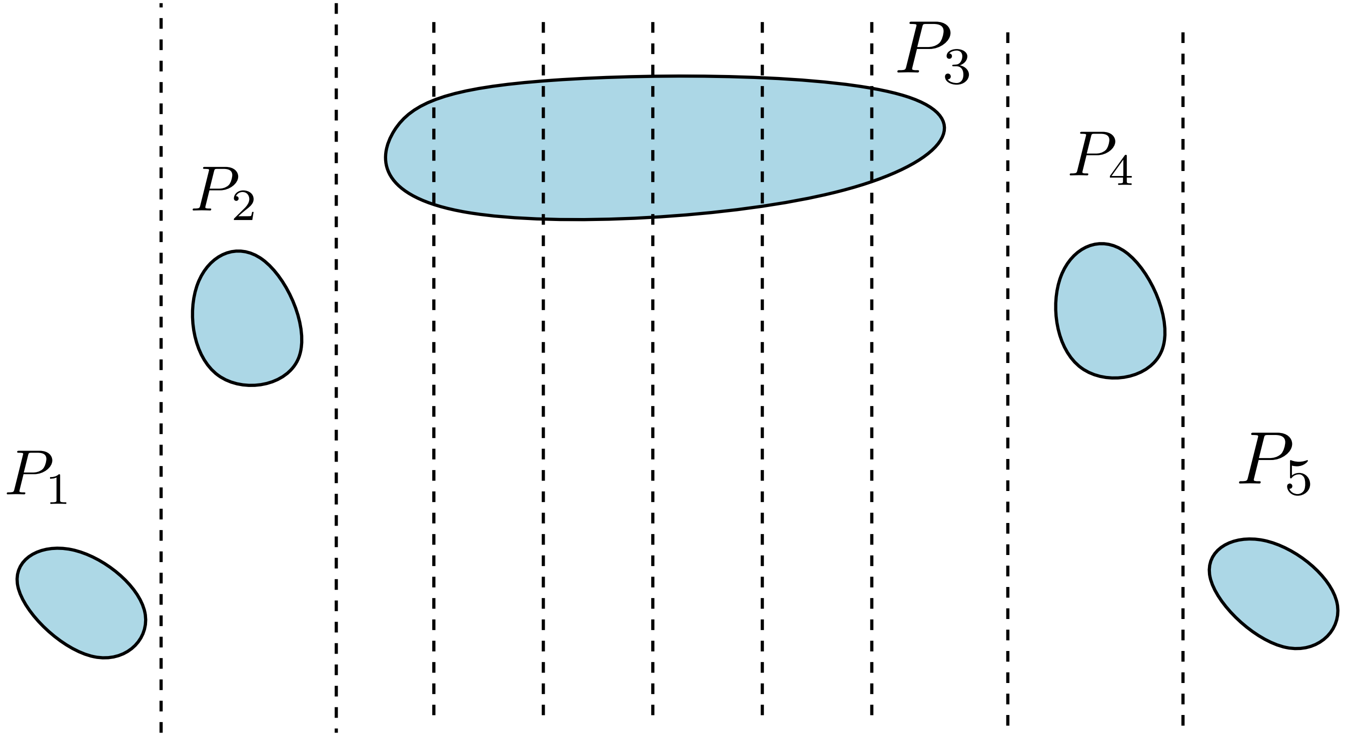

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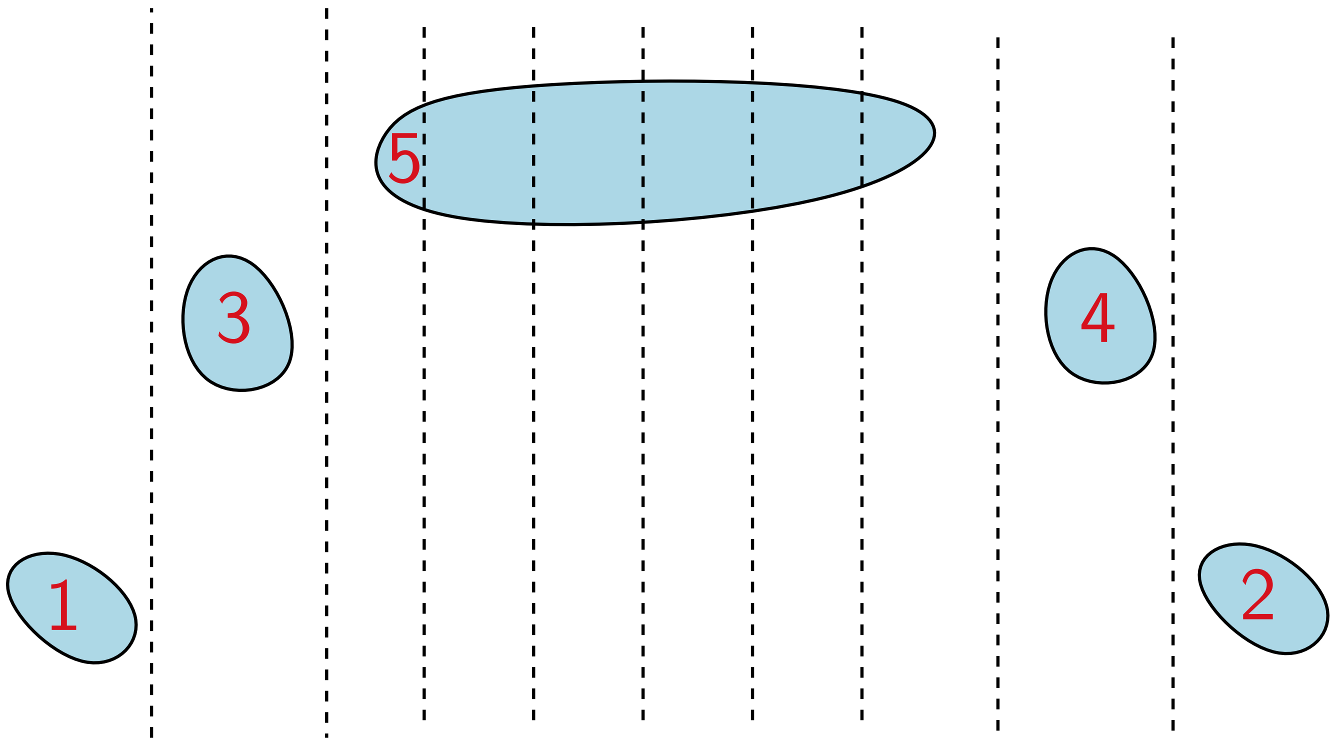

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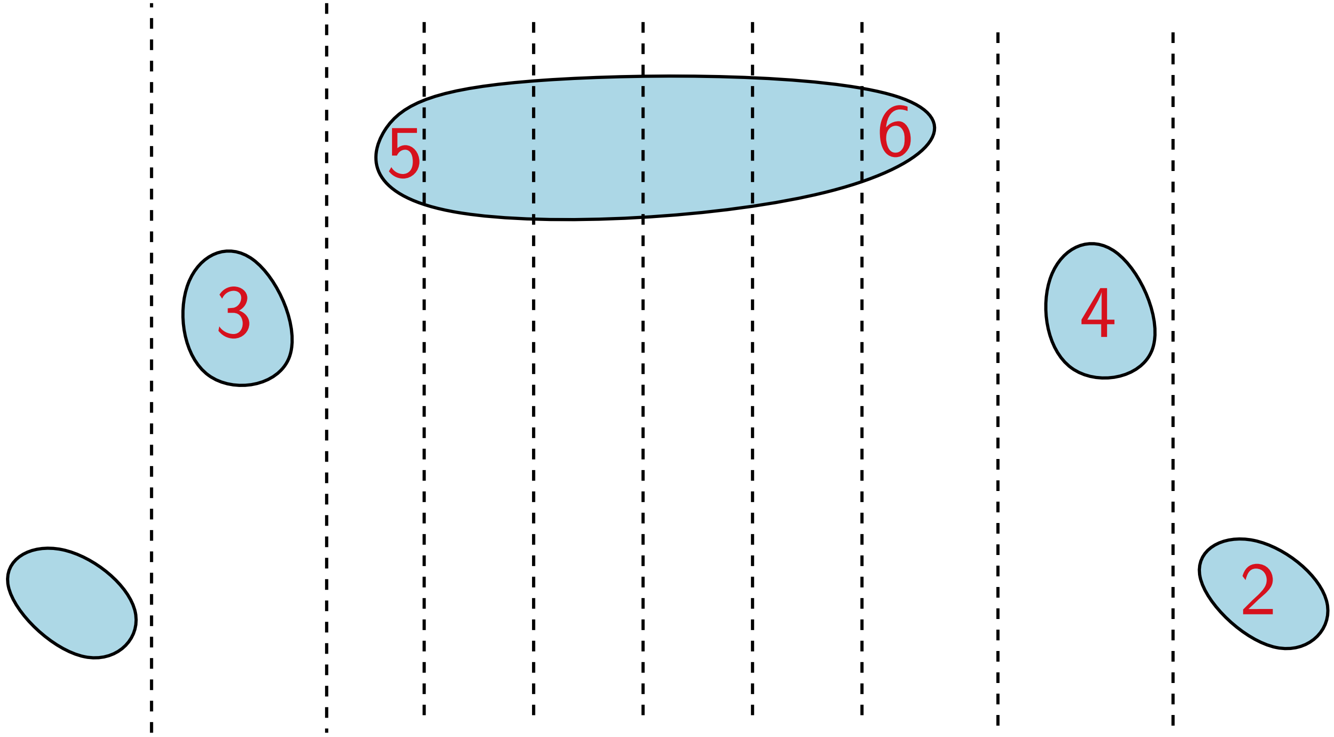

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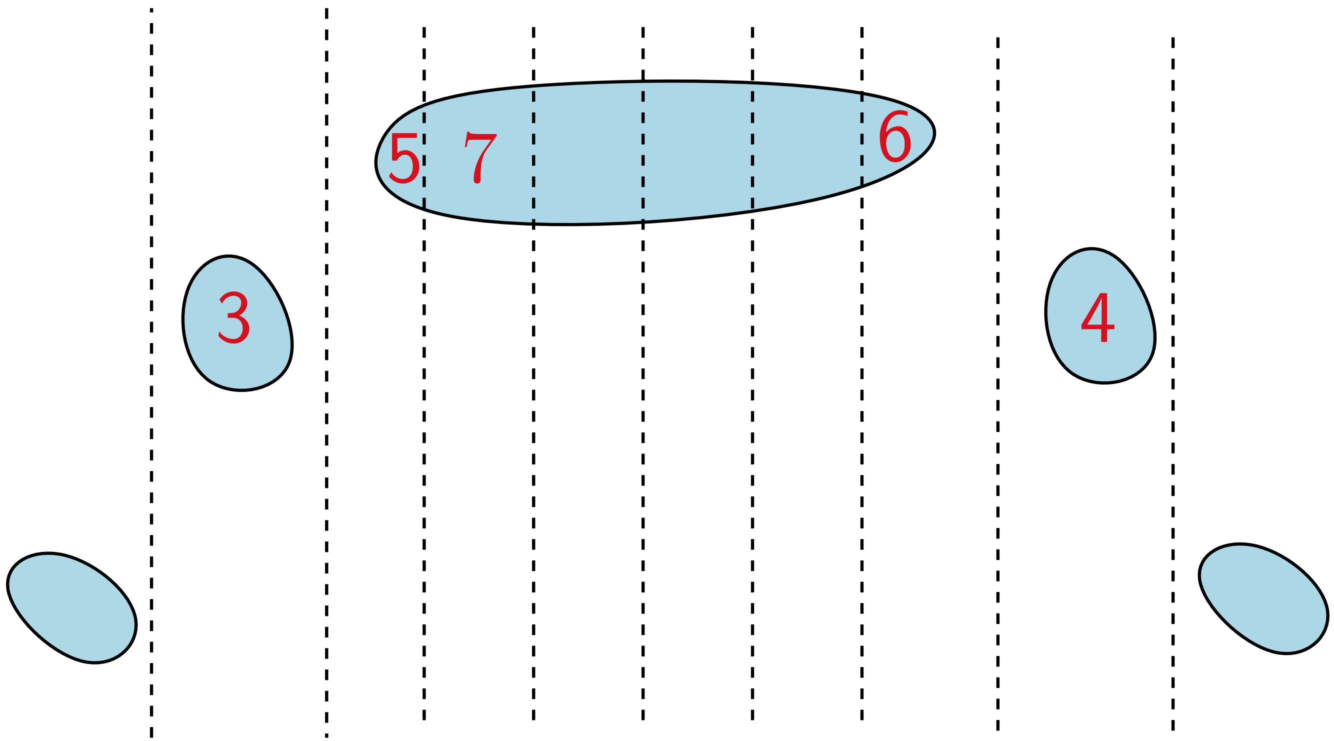

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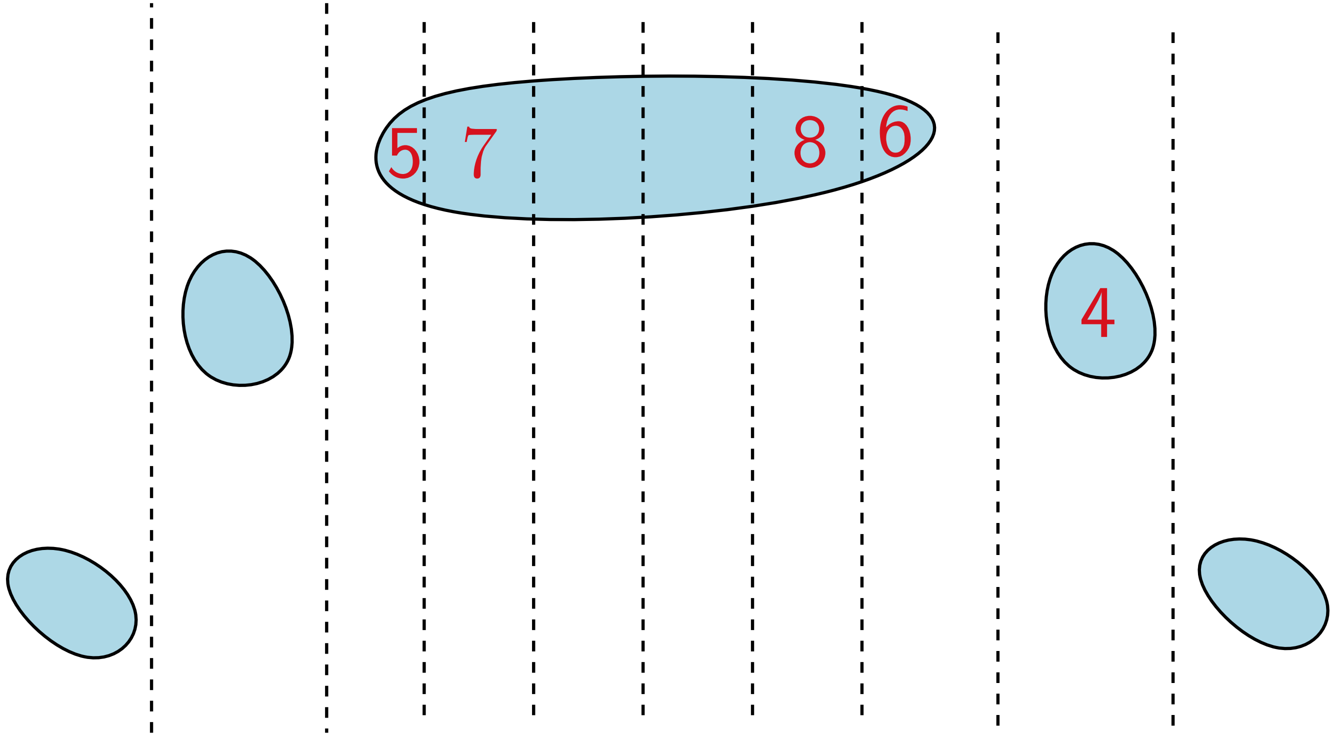

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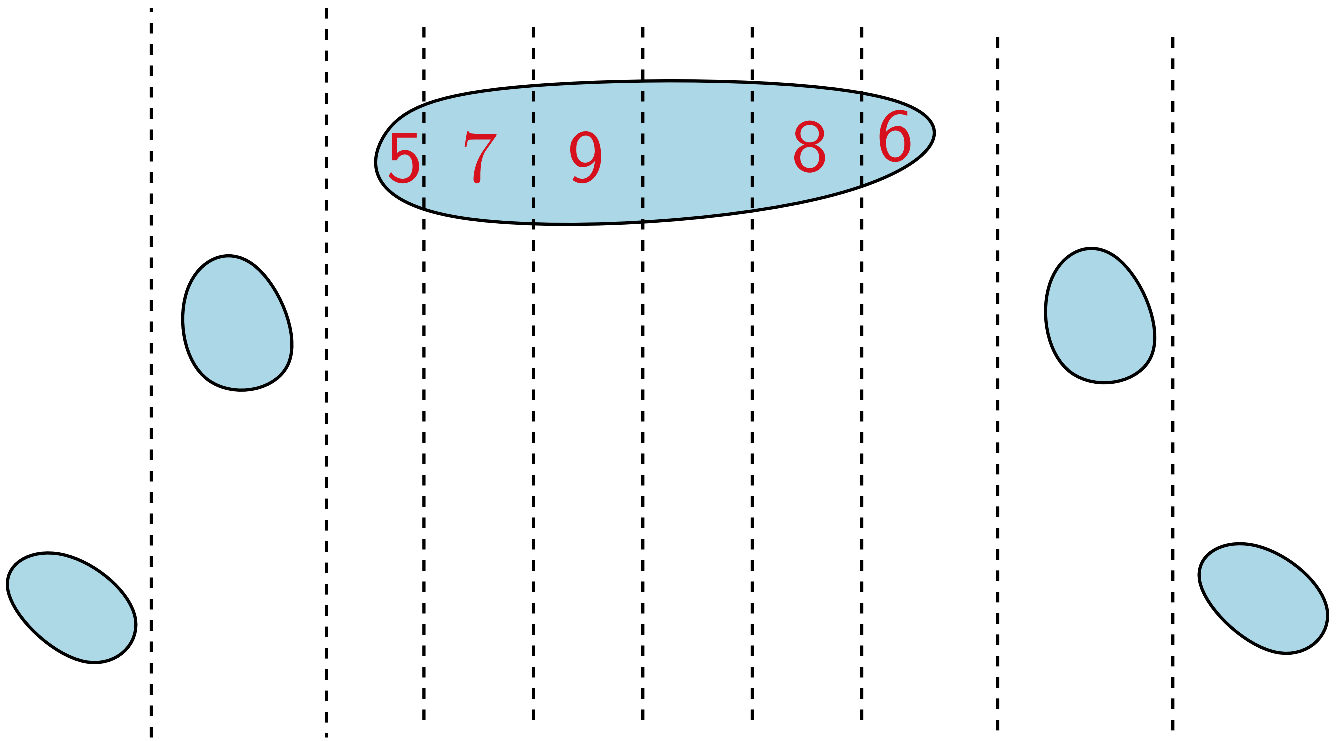

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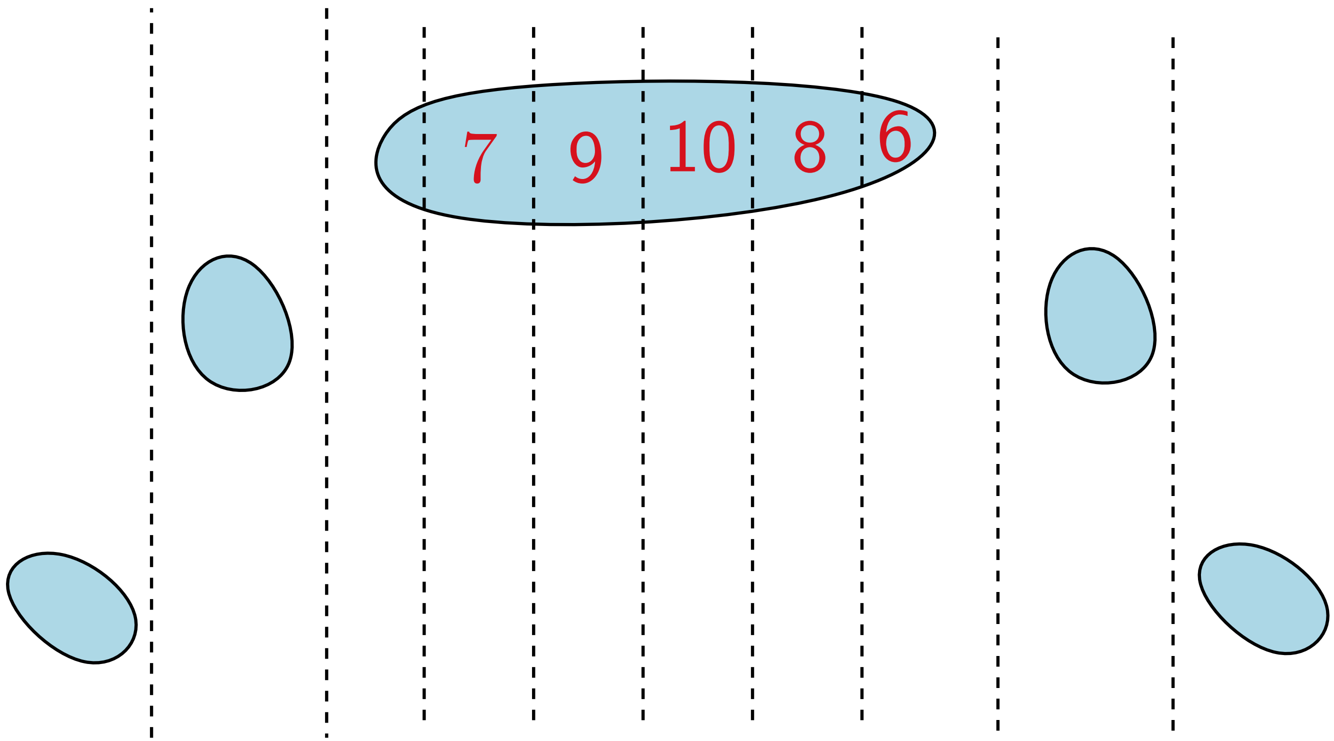

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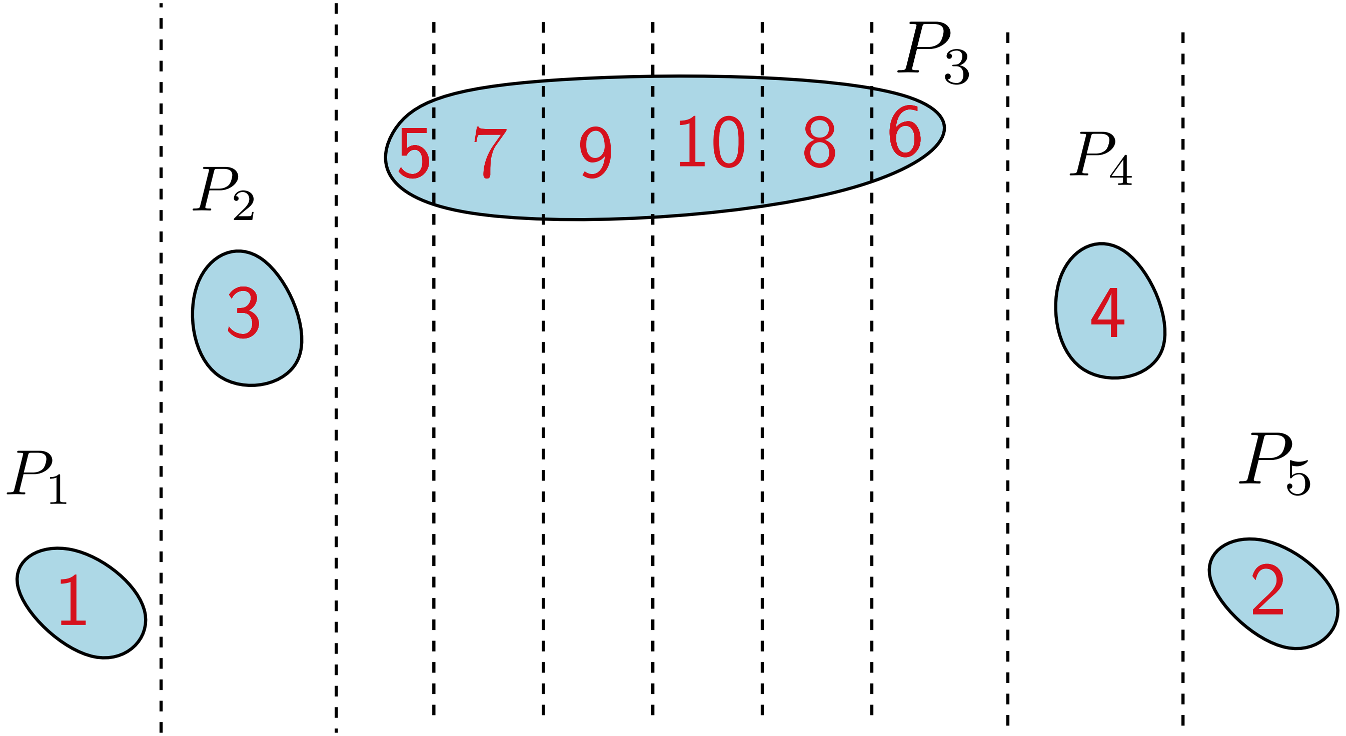

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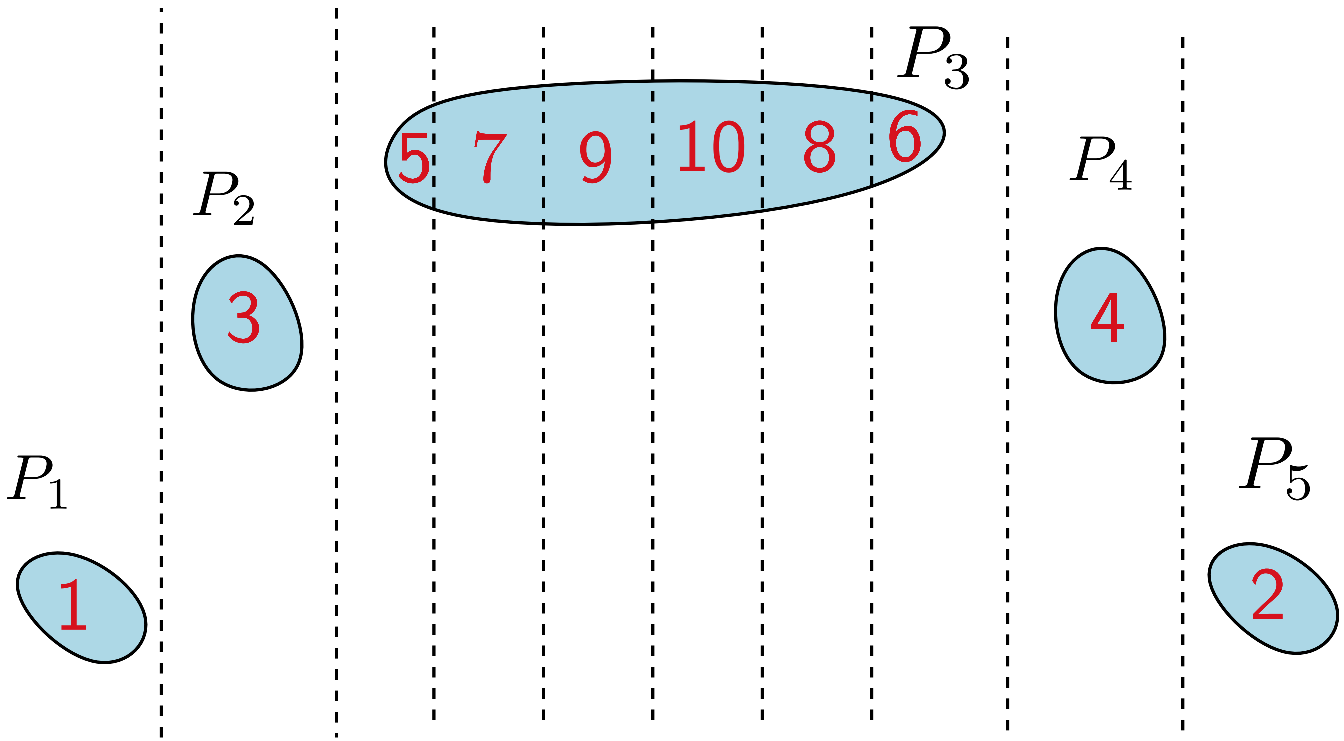

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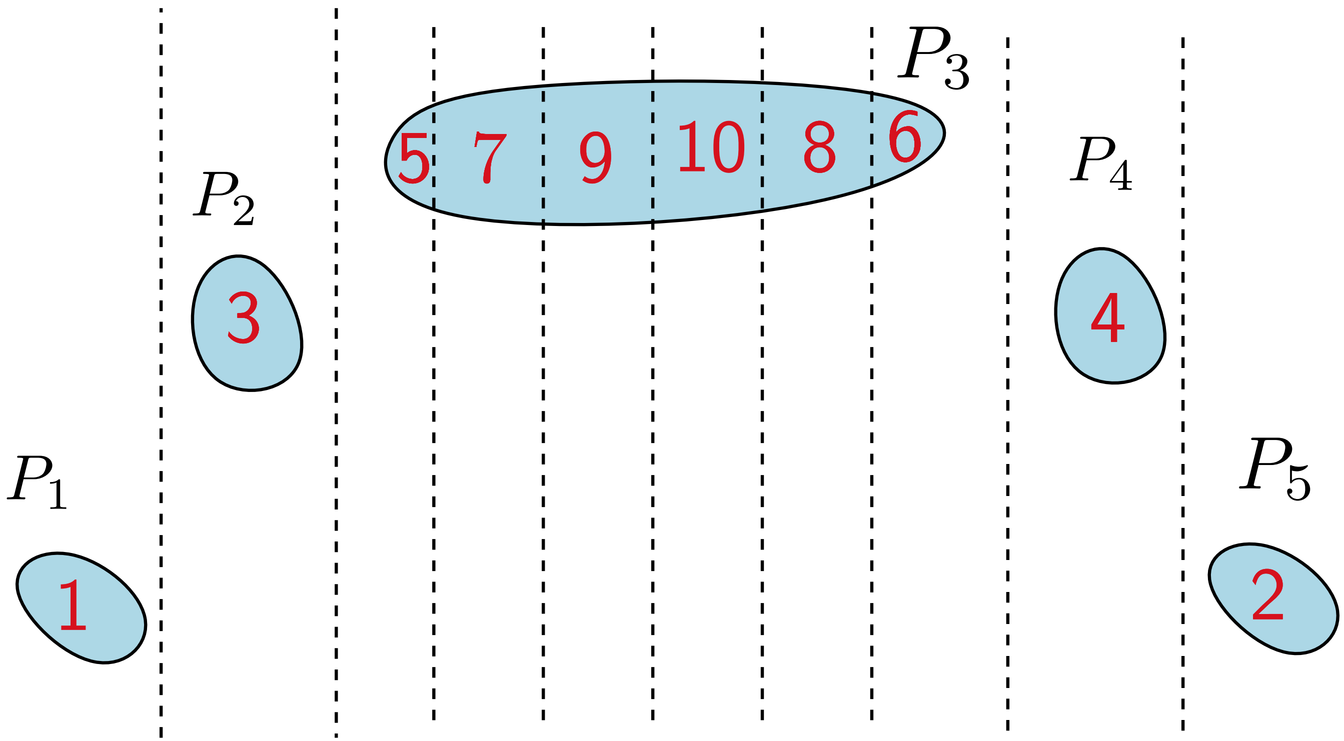

- Each blob in ≤ 5 5-tuples
- Recall, $|P| = Ckm$, $C = c(7)^{-1}c(5)^{-5}$
- We obtain sets $R_1, R_2, \ldots, R_k$ of size $> m$

Final step - sets 5-tuples form a cap

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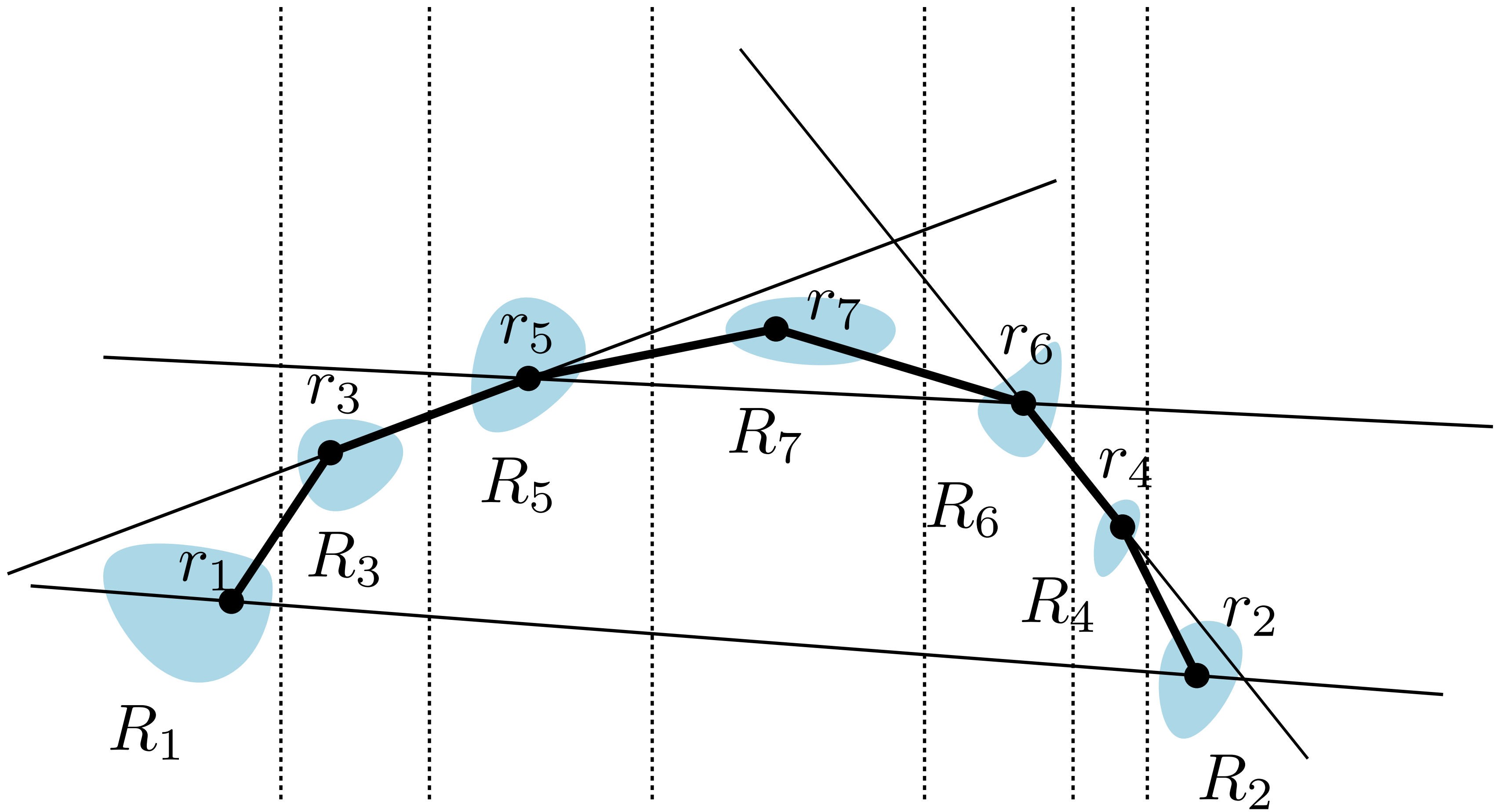

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Thm: For a set of n points P in general position, a crossing family of size $n/2^{O(\sqrt{\log m})}$ or a non-crossing family of size m can be computed in expected time $O\big(nm^{1+O((\log m)^{-1/3})}\big)$.

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Same type lemma is nondeterministic!

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Thank you for attention!