A Systematic Approach to Crossing Numbers of Cartesian Products with Paths

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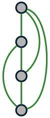
Cartesian Product of a Graph with a Path $G \square P_n$

Construction of $G \square P_n$

Example $K_4 \square P_4$



K_4

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copies $G^0, G^1, \ldots, G^n$ of G

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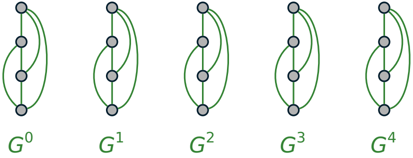

Cartesian Product of a Graph with a Path $G \Box P_n$

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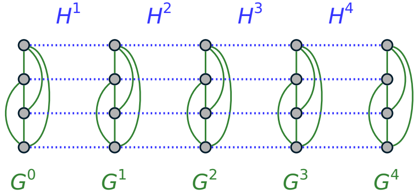

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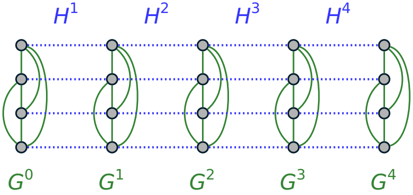

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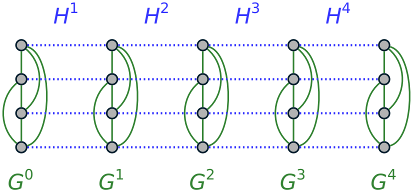

12 crossings, so $\mathrm{cr}(K_4 \square P_4) \leq 12$

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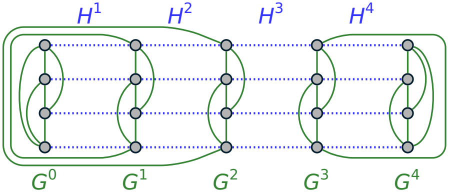

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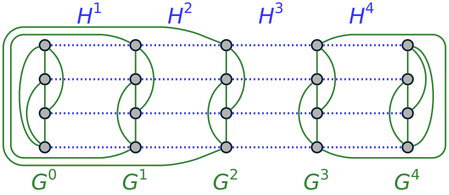

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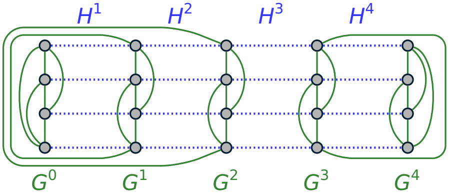
Our goal:

$\mathrm{cr}(G\square P_n)$ for a fixed graph G and general path length n

Example $K_4 \square P_4$



Canonical Drawings

vertices on a $(|V(G)| \times n)$-grid

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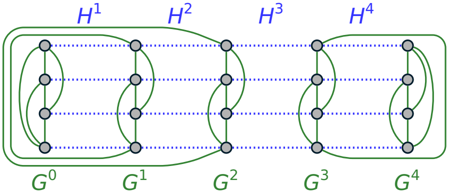

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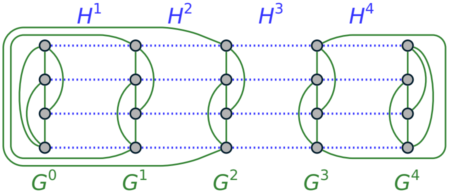

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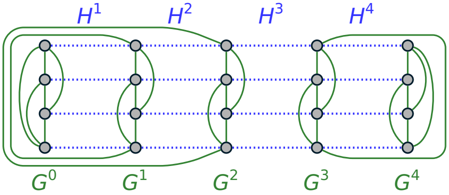

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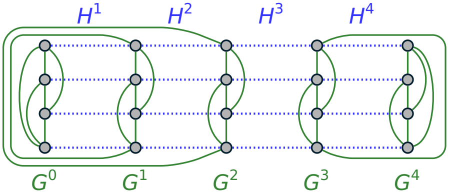

"middle" crossings are from
$G^i \times (H^i \cup H^{i+1})$

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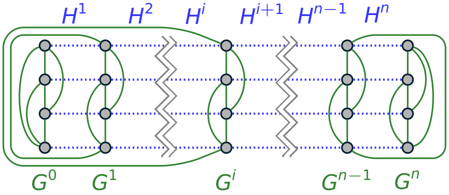

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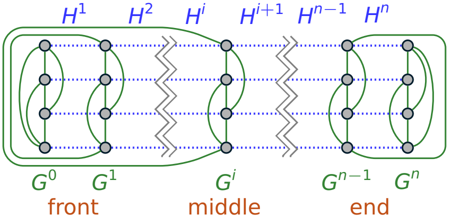

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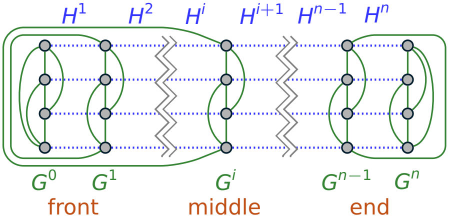
a crossings per middle copy

Example $K_4 \square P_n$ where $a=2$



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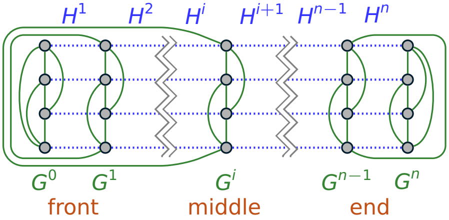
$an - b$ crossings in a canonical drawing

Example $K_4 \square P_n$ where $a{=}2, b{=}0$



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Upper Bound $\mathrm{cr}(G \square P_n) \le an - b$

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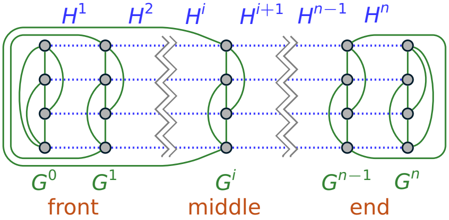

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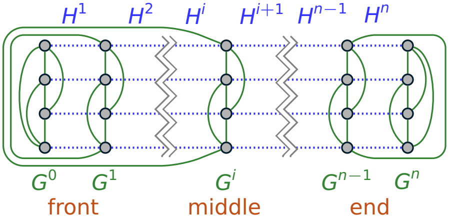

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Our goal: show $\mathrm{cr}(G \square P_n) = an - b$
(for fixed G, some $a, b \in \mathbb{N}$, and large enough n)

Lower Bound $\mathrm{cr}(G \square P_n) \geq an - b$

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~~copy edges are vertical (mostly)~~

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front, middle, and end copies

$\geq an - b$ crossings in any drawing

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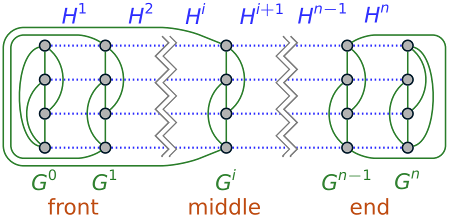

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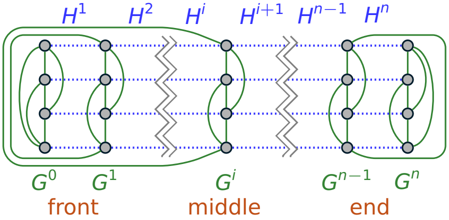

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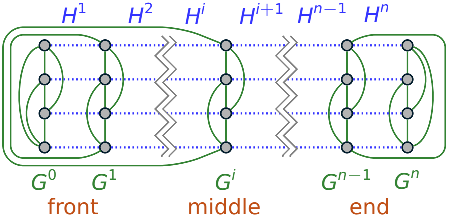

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“crossing bands”

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Known Results on $\mathrm{cr}(G \square P_n) = an - b$ (fixed G, some $a, b \in \mathbb{N}$ and large enough n)

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from optimal canonical drawing (or a known result for some $H \supset G$)

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successful for 96% of the 6-vertex and 79% of the 7-vertex graphs

a-Restricted Drawing

a-restricted drawing

each copy of G has fewer than a crossings on its edges

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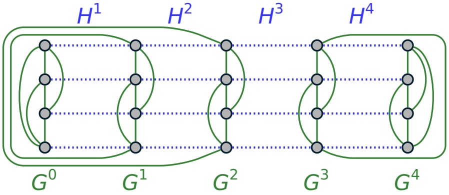

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Suppose $\mathrm{cr}(G \square P_n) < an - b$ **but** $\mathrm{cr}(G \square P_{n-1}) \geq a(n-1) - b$

consider a crossing-minimal drawing of $G \square P_n$

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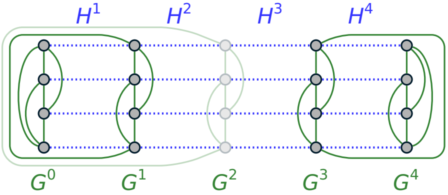

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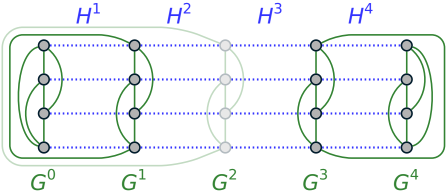

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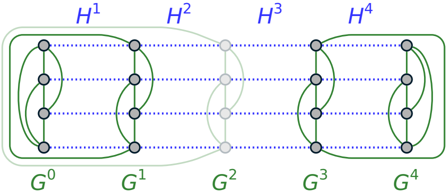

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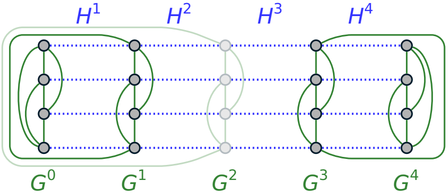

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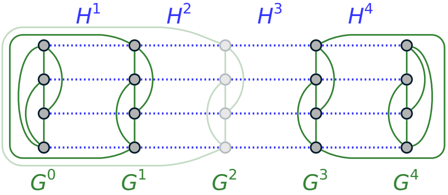

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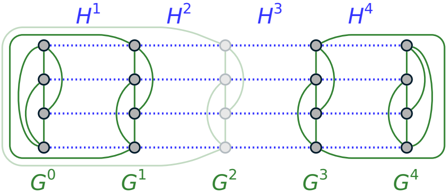

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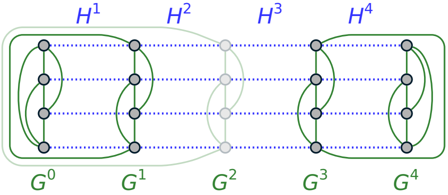

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from now on: only a-restricted drawings

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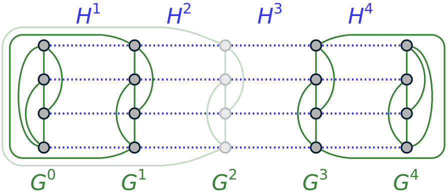

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need base case $\mathrm{cr}(G \square P_{n-1}) = a(n-1) - b$

Crossing Bands and Forces

Crossing band B_i

crossings that are “centered around G^i”

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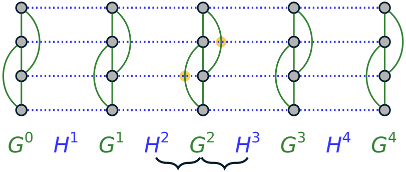

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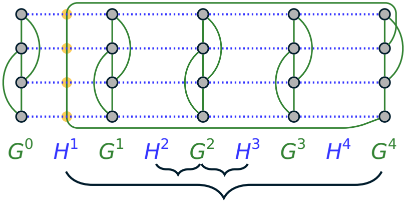

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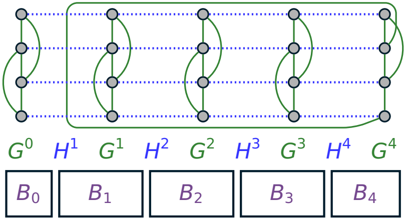

Crossing Bands and Forces

Crossing band B_i

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$\mathbf{cr}_a(G \square P_s, B_i \cup \cdots \cup B_j)$

min. # of crossings

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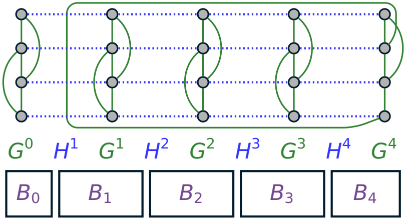

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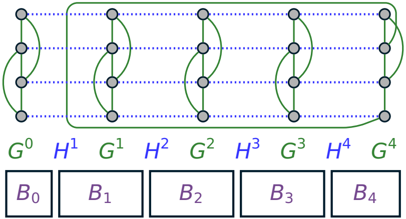

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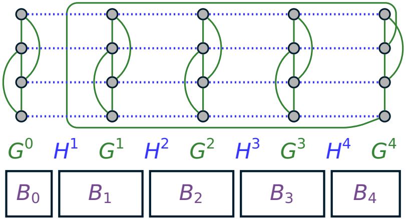

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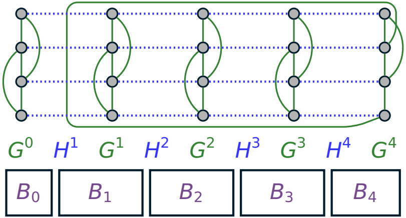

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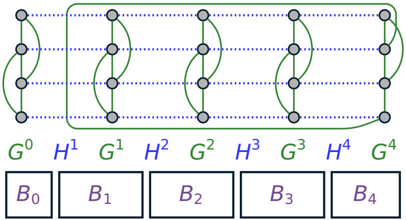

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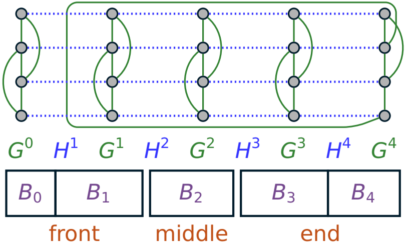

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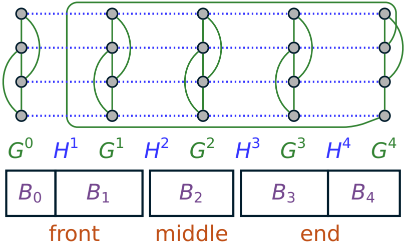

Forces

"min. # of front/middle/end crossings in any a-restricted drawing of $G\square P_s$"

partition crossing bands by middle index m

$$\begin{aligned}
\text{front force} \quad & \mathfrak{f}_{s,a}^{<m} && := \mathrm{cr}_a(G\square P_s, B_0 \cup \cdots \cup B_{m-1}) \\
\text{middle force} \quad & \mathfrak{f}_{s,a}^{m} && := \mathrm{cr}_a(G\square P_s, B_m) \\
\text{end force} \quad & \mathfrak{f}_{s,a}^{>m} && := \mathrm{cr}_a(G\square P_s, B_{m+1} \cup \cdots \cup B_s)
\end{aligned}$$

Lemma

In an a-restricted drawing of $G \square P_n$ there are

(a) *at least $\mathfrak{f}_{s,a}^{<m}$ crossings from the first m crossings bands;*

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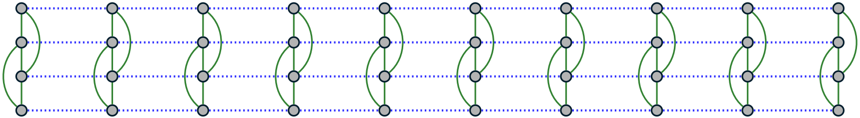
Example $K_4^-\square P_n$ where $n=9$, $a=2, b=0$



Forces in Action

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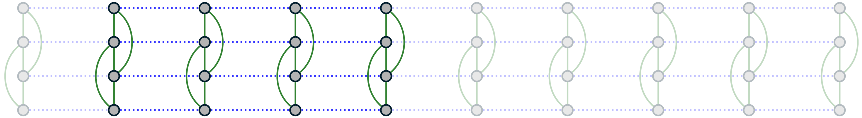

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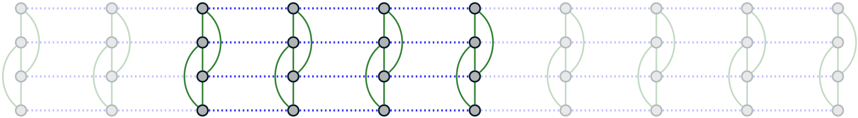

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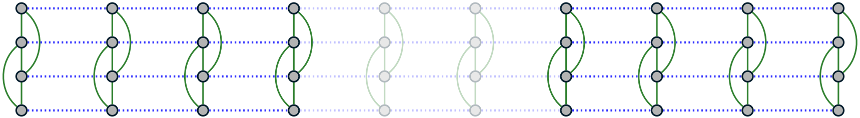

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no double counting by crossing band definition

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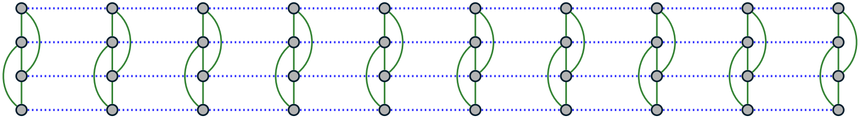

Main Theorem

forces: $\underbrace{\boxed{B_0}\boxed{\cdots}\boxed{B_{m-1}}}_{\mathfrak{f}^{<m}_{s,a}}$ $\underbrace{\boxed{B_m}}_{\mathfrak{f}^{m}_{s,a}}$ $\underbrace{\boxed{B_{m+1}}\boxed{\cdots}\boxed{B_s}}_{\mathfrak{f}^{>m}_{s,a}}$

Theorem

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, , *and* ,

(base case) *(middle)* *(front/end)*

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D_n: $\underbrace{\boxed{B_0}\boxed{\cdots}\boxed{B_{m-1}}}$ $\underbrace{\boxed{B_m}\boxed{B_{m+1}}\boxed{B_{m+2}}\boxed{\cdots}\boxed{B_{m+n-s-1}}\boxed{B_{m+n-s}}}$ $\underbrace{\boxed{B_{n-(s-m)+1}}\boxed{\cdots}\boxed{B_n}}$

front crossings (on m crossing bands) | middle crossings (on $n - s + 1$ crossing bands) | end crossings (on $s - m$ crossing bands)

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G has	$\mathfrak{f}^{<m}_{s,a}, \mathfrak{f}^{m}_{s,a}, \mathfrak{f}^{>m}_{s,a}$	total
5 vertices	19 90%	**21** **100**%
6 vertices	91 81%	**107** **96**%
7 vertices	529 62%	**676** **79**%

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Future

are there graphs for which canonical drawings are not optimal?

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application to crossing numbers of other graph products