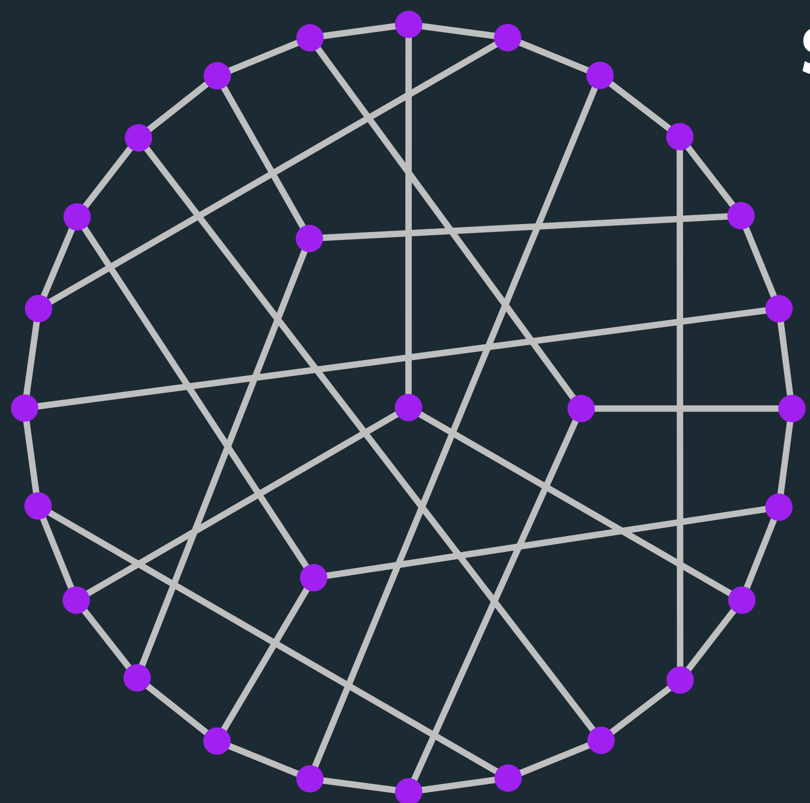
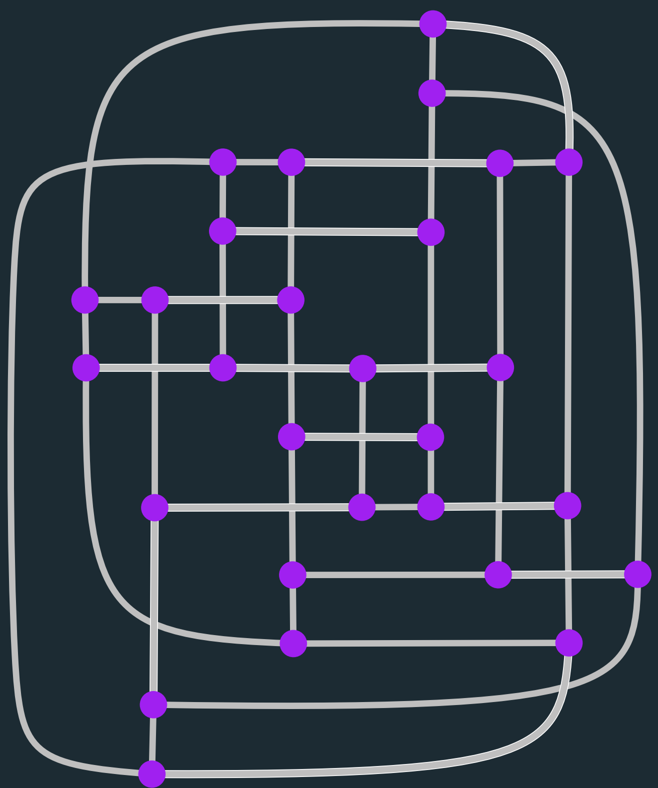
OOPS:

Optimized One-Planarity Solver via SAT

Sergey Pupyrev





Motivation

mathoverflow

Can a 3-regular non-1-planar graph be constructed?

Asked 2 years, 6 months ago Modified 8 months ago Viewed 1k times

21

A 1-planar graph is a graph which has a drawing on the plane such that each edge has at most one crossing.

I used `nauty` to generate all 3-regular graphs up to order 12, and checked each one of them individually. They all turned out to be 1-planar graphs.

My question is **whether it is possible to construct a 3-regular non-1-planar graph.**

Motivation

mathoverflow

Can a 3-regular non-1-planar graph be constructed?

Asked 2 years, 6 months ago Modified 8 months ago Viewed 1k times

21

A 1-planar graph is a graph which has a drawing on the plane such that each edge has at most one crossing.

I used `nauty` to generate all 3-regular graphs up to order 12, and checked each one of them individually. They all turned out to be 1-planar graphs.

My question is **whether it is possible to construct a 3-regular non-1-planar graph.**

4 Answers

Sorted by: Highest score (default)

16

For each t, there are 3-regular graphs that are not t-planar. In fact, a random 3-regular n-vertex graph, for large n, has this property. As we'll in the proof, one can take $n = Ct(\log t)^6$ for a suitable absolute constant C.

Motivation

11011110 About All posts

Cubic 1-planarity

May 11, 2014

Here's a graph drawing problem I don't know how to solve: find the smallest 3-regular graph that is not 1-planar.

The examples I tried with up to 24 vertices are 1-planar. Below are the 20-vertex Desargues graph,

•••

the 24-vertex McGee graph (7-cage),

•••

and the 24-vertex Nauru graph:

•••

On the other hand, for the 28-vertex Coxeter graph and 30-vertex Tutte–Coxeter graph (8-cage) I have been unable to find a 1-planar drawing. So my guess is that the answer is somewhere in this range.

Motivation

Question:
Which of the named WIKIPEDIA graphs are 1-planar?

WIKIPEDIA
The Free Encyclopedia
Search Wikipedia

List of graphs by edges and vertices

Article Talk

List [edit]

name	vertices	edges	radius	diam.
26-fullerene graph (26-fullerene)	26	39	5	6
120-cell	600	1200	15	15
Balaban 3-10-cage	70	105	6	6
Balaban 3-11-cage	112	168	6	8
Barnette–Bosák–Lederberg graph	38	57	5	9
Bidiakis cube	12	18	3	3
Biggs–Smith graph	102	153	7	7
Blanuša snarks	18	27	4	4

Stats:
- non-planar only
- $10 \leq |V| \leq 54$
- $15 \leq |E| \leq 90$
- **47** instances

Agenda

✓ 1. Motivation

→ 2. 8 attempts to build a solver

3. Some results

4. Open problems

Attempt **0 / 8**: Check Existing Work

Computational Geometry
Volume 108, January 2023, 101900

1-planarity testing and embedding: An experimental study ☆, ☆☆

Carla Binucci, Walter Didimo, Fabrizio Montecchiani

Show more

+ Add to Mendeley   Share   Cite

https://doi.org/10.1016/j.comgeo.2022.101900

no implementation available 😢

Attempt **0 / 8**: Check Existing Work

Computational Geometry
Volume 108, January 2023, 101900

1-planarity testing and embedding: An experimental study ☆, ☆☆

Carla Binucci, Walter Didimo, Fabrizio Montecchiani

Show more

Add to Mendeley   Share   Cite

https://doi.org/10.1016/j.comgeo.2022.101900   Get rights and content

no implementation available 😢

Current Results for Wikipedia:

- **0** 1-planar
- **0** non-1-planar
- **47** unknown

Attempt **1 / 8**: Brute-Force

Algorithm:

1. Identify pairs of potential crossing edges $O(m^2)$
2. Enumerate possible crossing pairs $O(2^{m^2})$
3. Check every combination for planarity $O(n \cdot 2^{m^2})$

Attempt **1 / 8**: Brute-Force

Algorithm:

1. Identify pairs of potential crossing edges $O(m^2)$
2. Enumerate possible crossing pairs $O(2^{m^2})$
3. Check every combination for planarity $O(n \cdot 2^{m^2})$

Current Results for Wikipedia:

- **11** 1-planar **+11**
- **0** non-1-planar
- **36** unknown

Attempt **2 / 8**: Lower Bounds

`Lemma 1:` [folklore]

A graphs is 1-planar iff every biconnected component is 1-planar.

`Lemma 2:` [Bodendiek, Schumacher, Wagner @ 1984]

Every 1-planar graph contains at most $4n - 8$ edges.

`Lemma 3:` [Karpov @ 2014]

Every **bipartite** 1-planar graph contains at most $3n - 8$ edges.

`Lemmas 4…:`

Every IC / NIC / girth-4 / …) 1-planar graph contains at most … edges.

Attempt **2 / 8**: Lower Bounds

`Lemma 1:` [folklore]

A graphs is 1-planar iff every biconnected component is 1-planar.

`Lemma 2:` [Bodendiek, Schumacher, Wagner @ 1984]

Every 1-planar graph contains at most $4n - 8$ edges.

`Lemma 3:` [Karpov @ 2014]

Every **bipartite** 1-planar graph contains at most $3n - 8$ edges.

`Lemmas 4...:`

Every IC / NIC / girth-4 / ...) 1-planar graph contains at most ... edges.

Current Results for Wikipedia:

- **12** 1-planar **+1**
- **0** non-1-planar
- **35** unknown

Attempt **3 / 8**: Naive SAT Encoding

Encode
Variables
Clauses

→

Solve
glucose
kissat
...

→

Result
YES / NO

Attempt **3 / 8**: Naive SAT Encoding

`Encode`
Variables
Clauses

`Solve`
glucose
kissat
...

`Result`
YES / NO

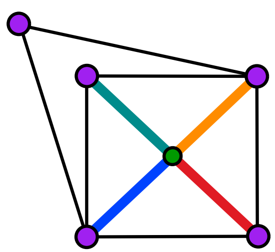
`Variables`

a pair of edges cross → 4 edges

an edge is uncrossed → 1 edge

`Clauses`

encoding planarity via
[Chimani & Zeranski @ 2014]
[Kirchweger, Scheucher, Szeider @ 2023]



Attempt **3 / 8**: Naive SAT Encoding

Encode
Variables
Clauses

→

Solve
glucose
kissat
...

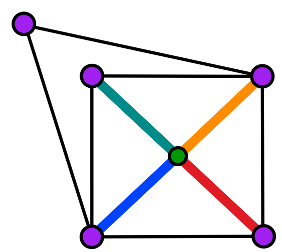
→

Result
YES / NO

Variables

a pair of edges cross → 4 edges

an edge is uncrossed → 1 edge



Clauses

encoding planarity via
[Chimani & Zeranski @ 2014]
[Kirchweger, Scheucher, Szeider @ 2023]

Current Results for Wikipedia:

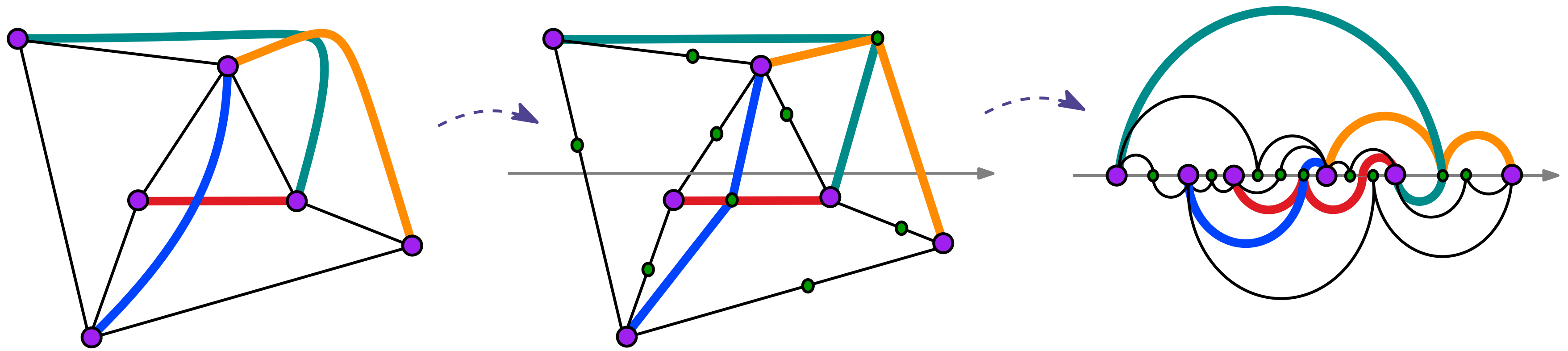
- **12** 1-planar
- **0** non-1-planar
- **35** unknown

Attempt **4 / 8**: Hanani-Tutte Encoding

inspired by:

[Chimani & Zeranski @ 2014]

[Fulek, Pelsmajer, Schaefer, Štefankovič @ 2013]

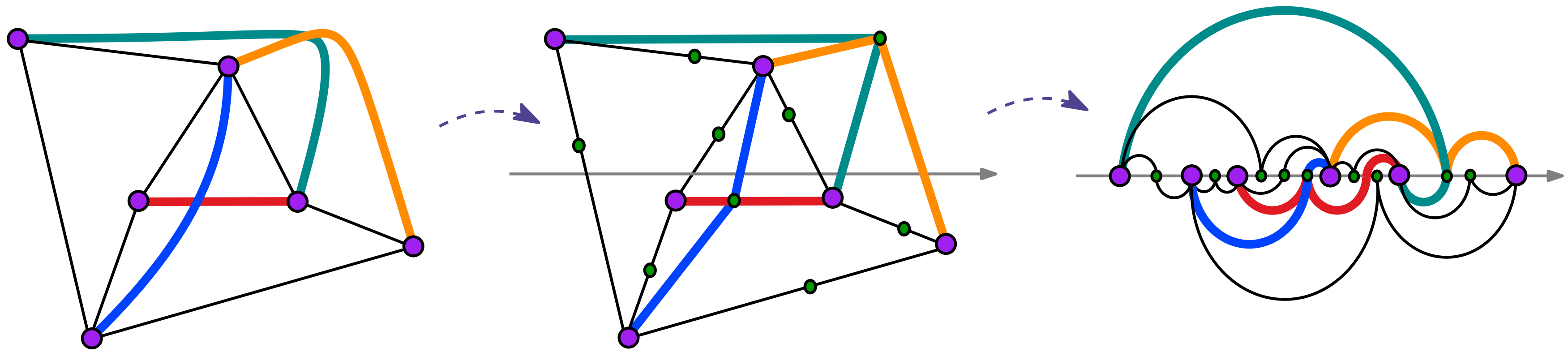


Attempt **4 / 8**: Hanani-Tutte Encoding

inspired by:

[Chimani & Zeranski @ 2014]

↳[Fulek, Pelsmajer, Schaefer, Štefankovič @ 2013]



Variables

- linear order of vertices
- pairs of crossing edges
- *move* variables: an edge is routed above/below a vertex

Clauses

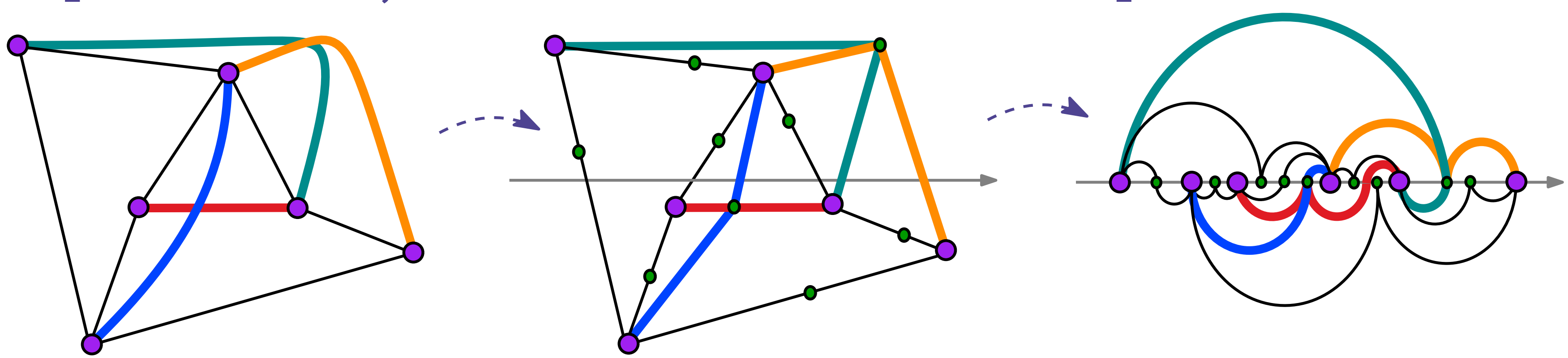
- sync the order & crossings & *move*
- *move* "planarity" constraints

Attempt **4 / 8**: Hanani-Tutte Encoding

inspired by:

[Chimani & Zeranski @ 2014]

↳[Fulek, Pelsmajer, Schaefer, Štefankovič @ 2013]



Variables

- linear order of vertices
- pairs of crossing edges
- *move* variables: an edge is routed above/below a vertex

Clauses

- sync the order & crossings & *move*
- *move* “planarity” constraints

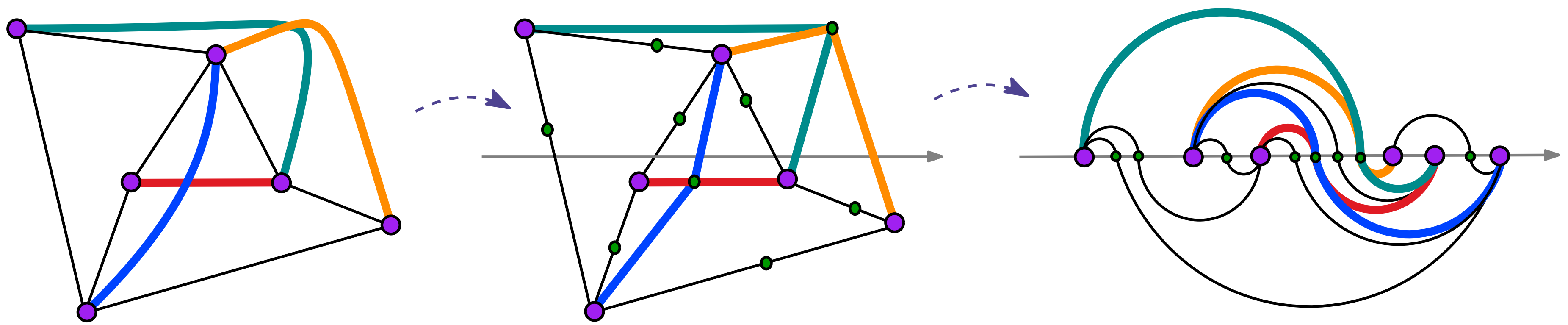
Current Results for Wikipedia:

- **26** 1-planar **+14**
- **0** non-1-planar
- **21** unknown

Attempt **5 / 8**: Book Embedding Encoding

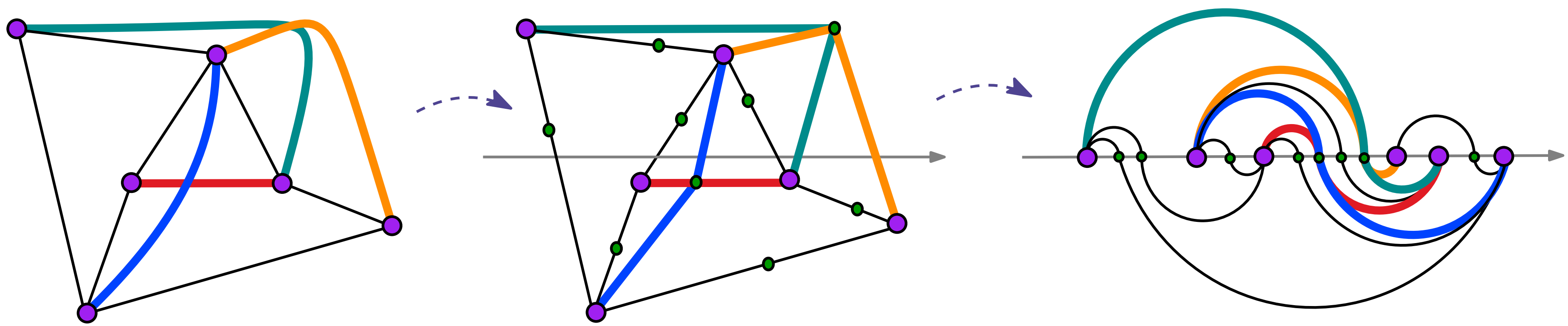
inspired by:

[Felsner, Fusy, Noy, Orden @ 2011]



Attempt **5 / 8**: Book Embedding Encoding

inspired by:
[Felsner, Fusy, Noy, Orden @ 2011]



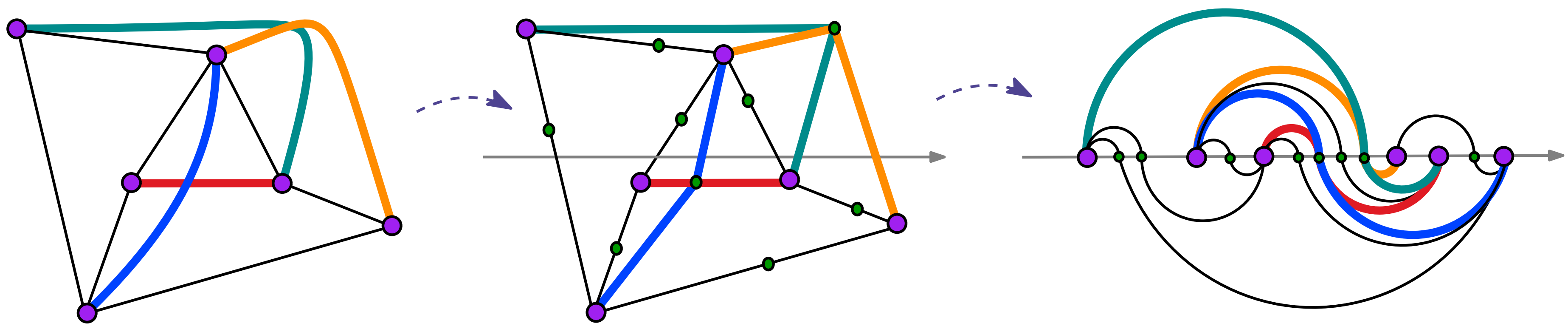
Variables
- linear order of vertices
- pairs of crossing edges
- edge above/below the spine

Clauses
- sync the order & crossings
- planarity constraints

Attempt **5 / 8**: Book Embedding Encoding

inspired by:

[Felsner, Fusy, Noy, Orden @ 2011]



Variables

- linear order of vertices
- pairs of crossing edges
- edge above/below the spine

Clauses

- sync the order & crossings
- planarity constraints

Current Results for Wikipedia:

- **29** 1-planar **+3**
- **1** non-1-planar **+1**
- **17** unknown

Attempt **6 / 8**: Reduce Search Space

Ad-hoc

Variables

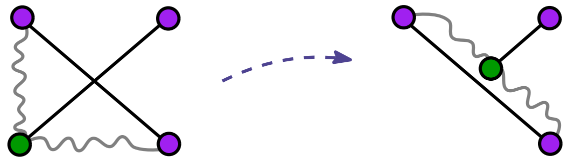
- linear order of vertices n^2
- pairs of crossing edges m^2
- edge above/below the spine m

Automatic

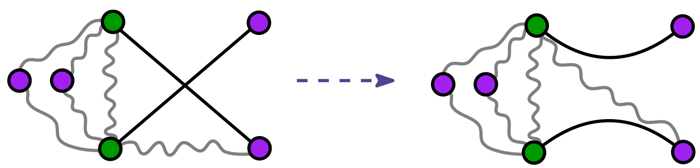
Attempt **6 / 8**: Reduce Search Space

Ad-hoc

Rule 1:



Rule 2:



...

Automatic

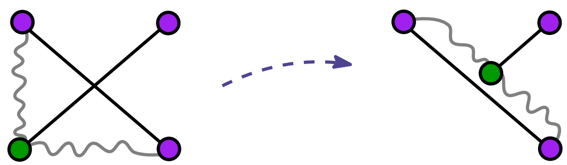
Variables

- linear order of vertices n^2
- pairs of crossing edges m^2
- edge above/below the spine m

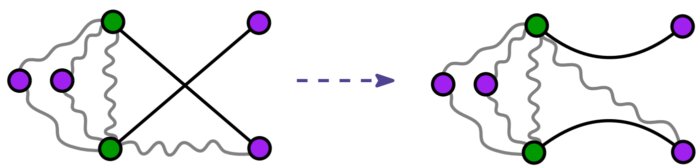
Attempt **6 / 8**: Reduce Search Space

Ad-hoc

Rule 1:



Rule 2:

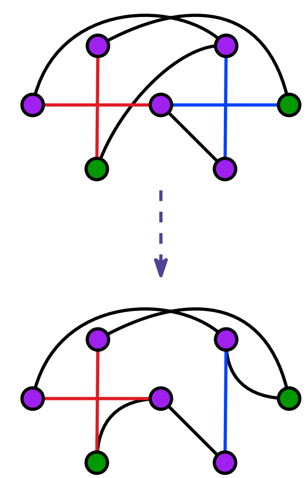


...

Variables

- linear order of vertices n^2
- pairs of crossing edges m^2
- edge above/below the spine m

Automatic

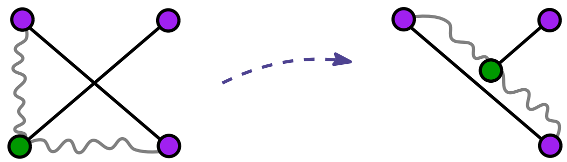


1. Enumerate all triples of crossing pairs
2. Check if "rerouting" eliminates a crossing
3. If yes, introduce a 3-clause

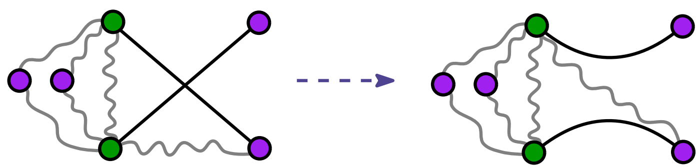
Attempt **6 / 8**: Reduce Search Space

Ad-hoc

Rule 1:



Rule 2:



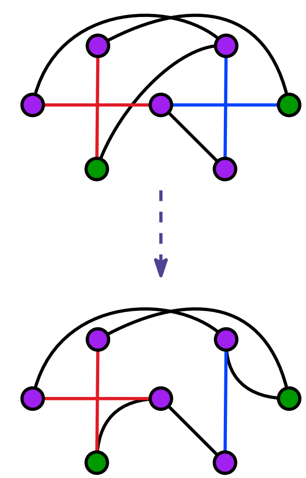
...

Variables

- linear order of vertices n^2
- pairs of crossing edges m^2
- edge above/below the spine m

Automatic

1. Enumerate all triples of crossing pairs
2. Check if "rerouting" eliminates a crossing
3. If yes, introduce a 3-clause



Current Results for Wikipedia:

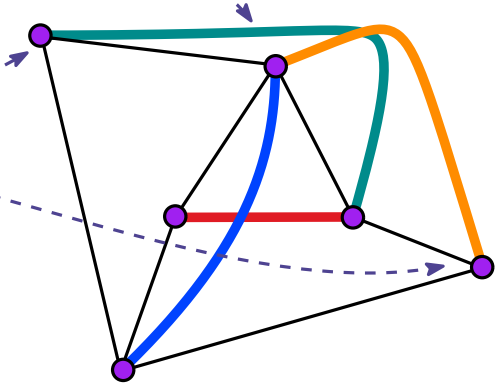
- **29** 1-planar
- **4** non-1-planar **+3**
- **14** unknown

Attempt **7 / 8**: Break Symmetries

`Ad-hoc`

- first vertex in the order
- relative order of twins

...



`Automatic`

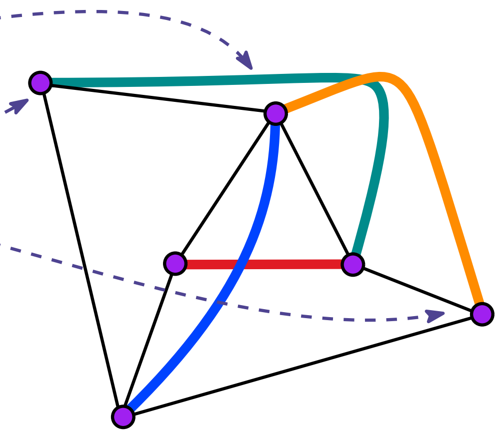
- **`https://github.com/meelgroup/breakid`** [Devriendt and Bogaerts @ 2025]
- **`https://github.com/markusa4/satsuma`** [Anders, Brenner, Rattan @ 2024]

Attempt **7 / 8**: Break Symmetries

Ad-hoc

- first vertex in the order
- relative order of twins

...



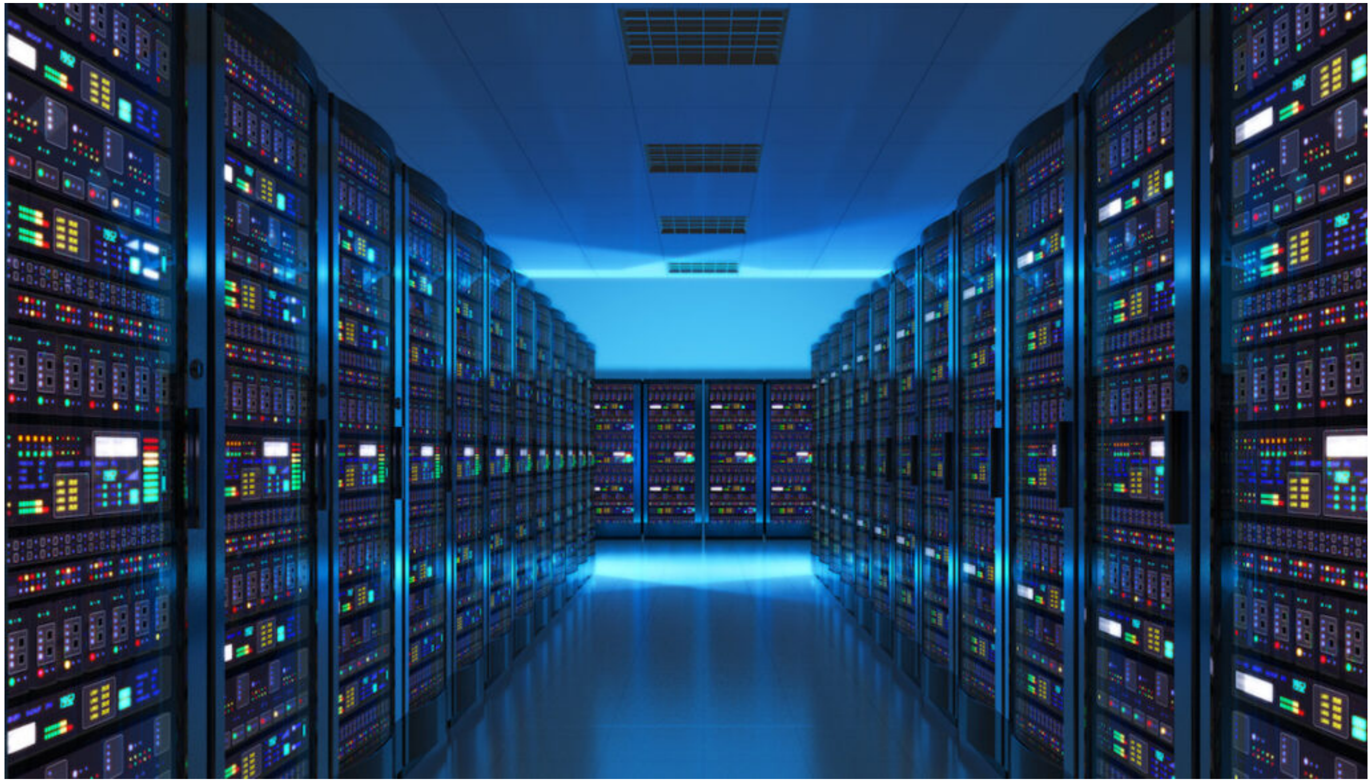
Automatic

- **https://github.com/meelgroup/breakid** [Devriendt and Bogaerts @ 2025]
- **https://github.com/markusa4/satsuma** [Anders, Brenner, Rattan @ 2024]

Current Results for Wikipedia:

- **29** 1-planar
- **5** non-1-planar **+1**
- **13** unknown

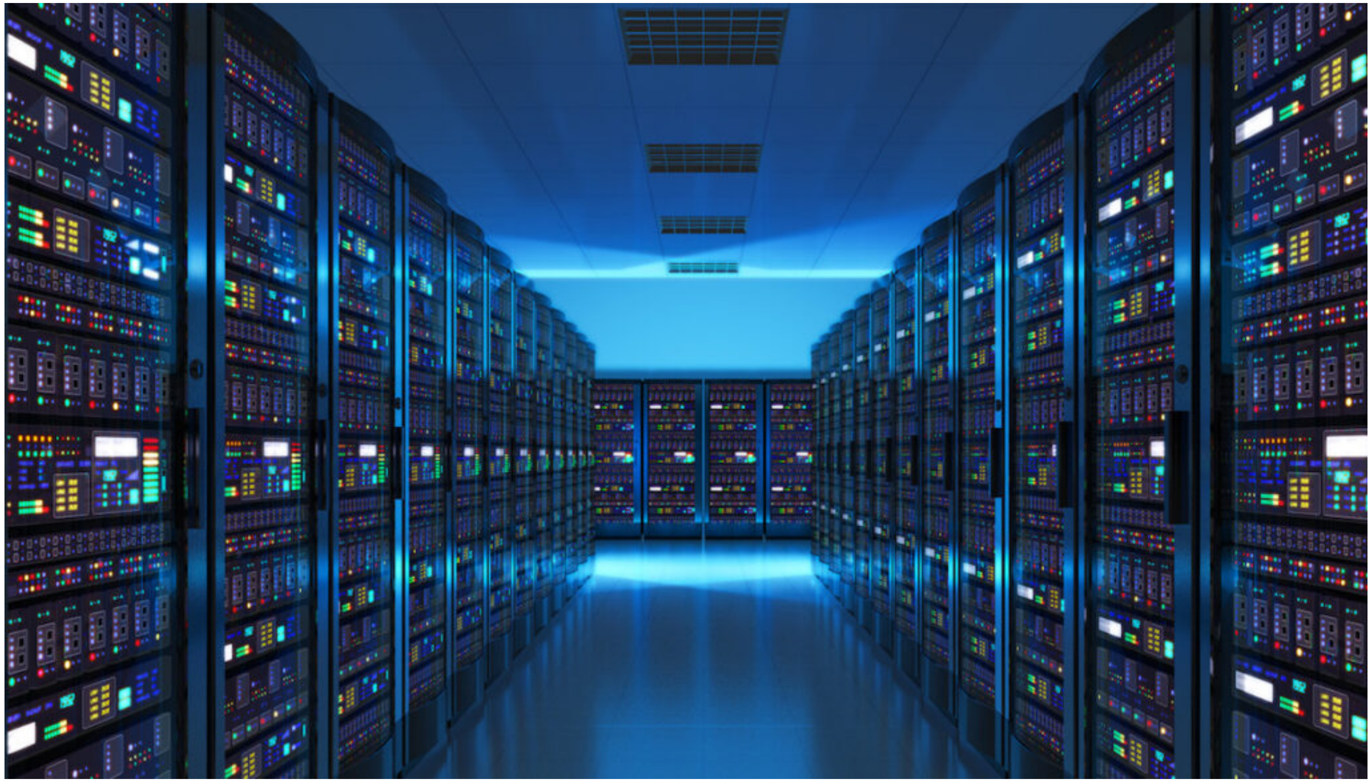
Attempt **8 / 8**: Brute-Scale



parallel
SAT Solvers:

- GIMSATUL
- GLUCOSE-24/48
- PARKISSAT
- PRS
- **PAINLESS**
- MALLOB
- LINGELING
- ...

Attempt **8 / 8**: Brute-Scale



parallel SAT Solvers:

- GIMSATUL
- GLUCOSE-24/48
- PARKISSAT
- PRS
- **PAINLESS**
- MALLOB
- LINGELING
- ...

Current Results for WIKIPEDIA:

- **29** 1-planar
- **15** non-1-planar **+10**
- **3** unknown

Agenda

✓ 1. Motivation

✓ 2. 8 attempts to build a solver

→ 3. Some results

4. Open problems

Results (practical)

- OOPS is open sourced at https://github.com/spupyrev/oops

most	*small*	instances ($n \leq 30, m \leq 50$):	(milli-)seconds
many	*medium*	instances ($n \leq 60, m \leq 75$):	minutes
some	*large*	instances ($n \leq 90, m \leq 100$):	hours

1-planar instances recognized **much** faster than non-1-planar: minutes vs days

Results (practical)

- OOPS is open sourced at https://github.com/spupyrev/oops

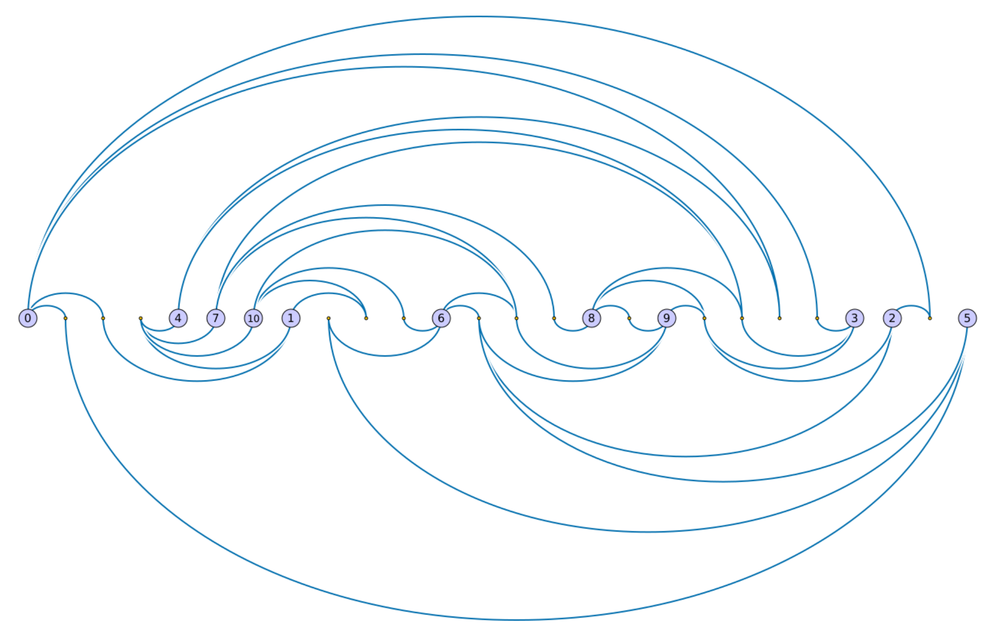
  most *small* instances ($n \leq 30, m \leq 50$): (milli-)seconds

  many *medium* instances ($n \leq 60, m \leq 75$): minutes

  some *large* instances ($n \leq 90, m \leq 100$): hours

  1-planar instances recognized **much** faster than non-1-planar: minutes vs days

- misc extensions:
  - `./oops -ic`
  - `./oops -nic`
  - `./oops -directed`
  - `./oops -output=svg`



Results (theoretical)

Theorem 1 (computationally verified)

Every graph with $n \leq 6$ or $m \leq 17$ is 1-planar.

Every **bipartite** graph with $n \leq 8$ or $m \leq 18$ is 1-planar.

Theorem 2

There exist non-1-planar graphs with $m = n + 10$.

($m = n + 3$ is the tight *planar* bound)

Results (theoretical)

Theorem 1 (computationally verified)

Every graph with $n \leq 6$ or $m \leq 17$ is 1-planar.

Every **bipartite** graph with $n \leq 8$ or $m \leq 18$ is 1-planar.

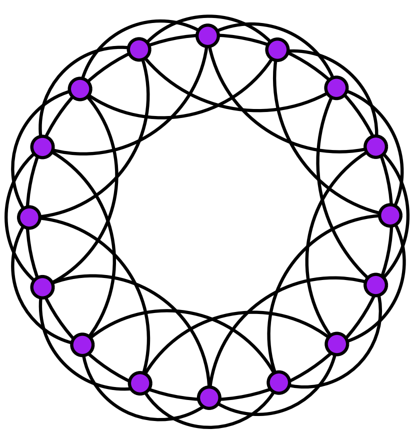
Theorem 2

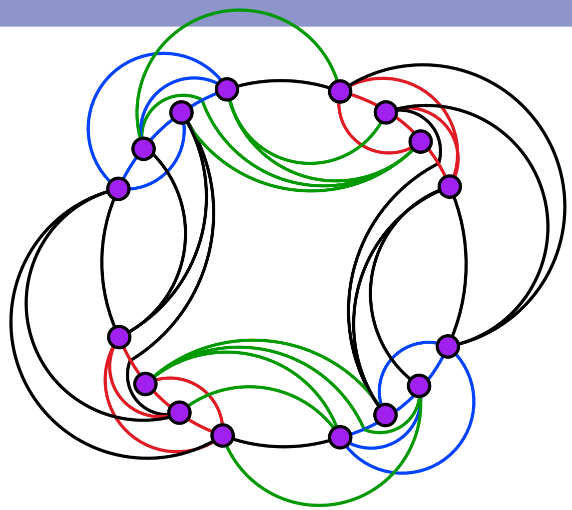
There exist non-1-planar graphs with $m = n + 10$.

($m = n + 3$ is the tight *planar* bound)

Theorem 3 answers a question in [Zhang, Ouyang, Huang'25]

There exist infinitely many 6-connected claw-free 1-planar graphs.





Future

Problem 1

Find the smallest cubic non-1-planar graph

Future

Problem 1

Find the smallest cubic non-1-planar graph

Conjecture: TUTTE-COXETER with $n = 30$, along with 4 other instances (verified for all $n \leq 24$ and all bipartite $n \leq 30$)

Future

Problem 1

Find the smallest cubic non-1-planar graph

Conjecture: TUTTE-COXETER with $n = 30$, along with 4 other instances (verified for all $n \leq 24$ and all bipartite $n \leq 30$)

Problem 2

Design a **formal proof** of such a result (and other results in GD) [in Lean?]

Future

Problem 1
Find the smallest cubic non-1-planar graph

Conjecture: Tutte-Coxeter with $n = 30$, along with 4 other instances
(verified for all $n \leq 24$ and all bipartite $n \leq 30$)

Problem 2
Design a **formal proof** of such a result (and other results in GD) [in Lean?]

Problem 3
Is it NP-hard to recognize **cubic** 1-planar graphs?

Problem 4
Design SAT solvers for 2-planar, fan-planar, **straight-line** 1-planar, ... graphs

Future

Problem 1

Find the smallest cubic non-1-planar graph

Conjecture: Tutte-Coxeter with $n = 30$, along with 4 other instances (verified for all $n \leq 24$ and all bipartite $n \leq 30$)

Problem 2

Design a **formal proof** of such a result (and other results in GD) [in Lean?]

Problem 3

Is it NP-hard to recognize **cubic** 1-planar graphs?

Problem 4

Design SAT solvers for 2-planar, fan-planar, **straight-line** 1-planar, ... graphs