Characterizing and Recognizing Twistedness

Oswin Aichholzer TU Graz

Alfredo García Universidad Zaragoza 1474

Javier Tejel Universidad Zaragoza 1474

Birgit Vogtenhuber TU Graz

Alexandra Weinberger FernUniversität in Hagen

Characterizing and Recognizing Twistedness

An abstract rotation system of K_n with $n \geq 7$ is generalized twisted
$\Leftrightarrow$ every subrotation system induced by 5 vertices is.

A realizable rotation system of K_n with $n \geq 3$ is generalized twisted $\Leftrightarrow$
there exist two vertices v_1 and v_2 and a bipartition $A \cup B$ of the remaining vertices (A or B can be empty) s.t.:

- $\forall$ a, a' in A, the edge aa' crosses v_1v_2.
- $\forall$ b, b' in B, the edge bb' crosses v_1v_2.
- $\forall$ $a \in A$, $b \in B$, the edge ab does not cross v_1v_2.
- Rotation of v_1: Beginning at v_2, all vertices in B appear before all vertices in A.
- Rotation of v_2: Beginning at v_1, all vertices in B appear before all vertices in A.

Realizable rotation system of K_n: $O(n^2)$-time algorithm to decide if generalized twisted
Abstract rotation system of K_n: $O(n^5)$-time algorithm to decide if generalized twisted

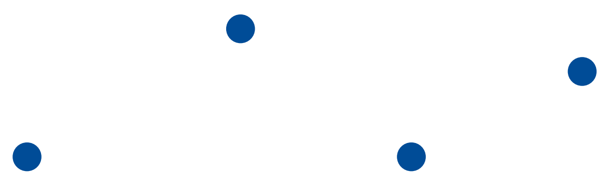
Characterizing and Recognizing Twistedness

- Introduction and definitions
  (simple drawings, generalized twisted drawings, rotation systems)
- Characterizing and recognizing generalized twisted drawings
  (Combinatorial characterization via small subdrawings and via a 'special' vertex pair leading to fast recognition algorithms)
- Most important parts of the algorithm

Simple Drawings

Simple drawings (simple topological graphs, good drawings) are drawings of graphs on the plane, where:

- Vertices are disjoint points, edges are Jordan arcs.



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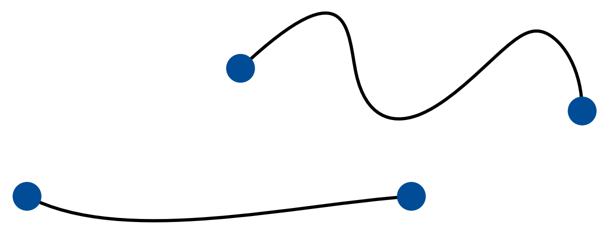

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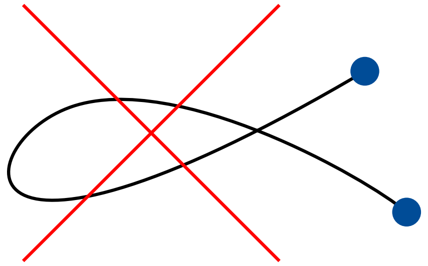

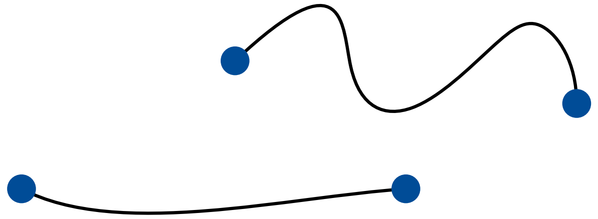

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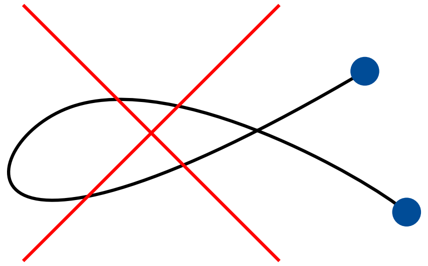
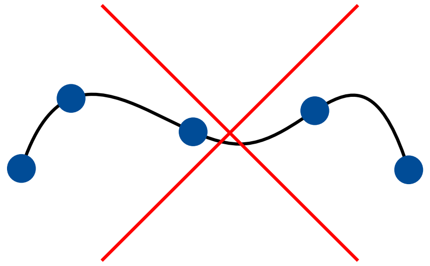
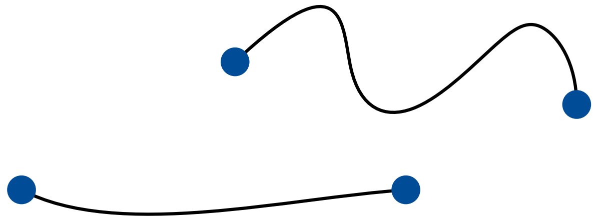


- Edges don't pass through other vertices.





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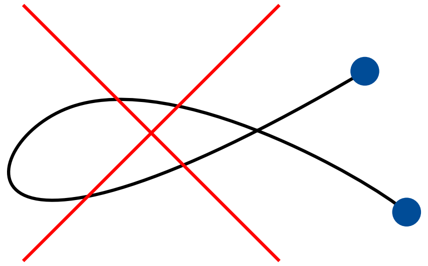

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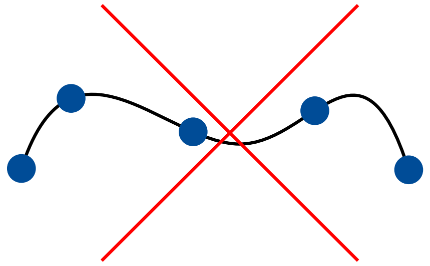

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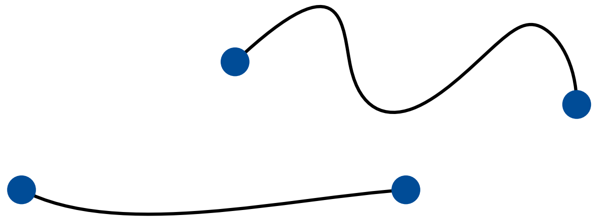
  No adjacent edges cross, no multiple crossings!



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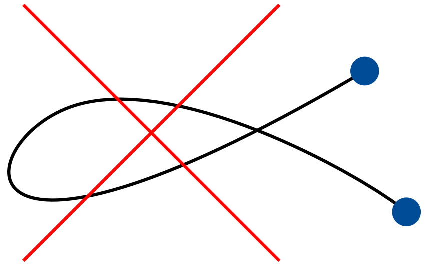

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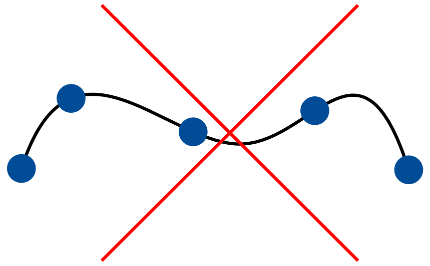

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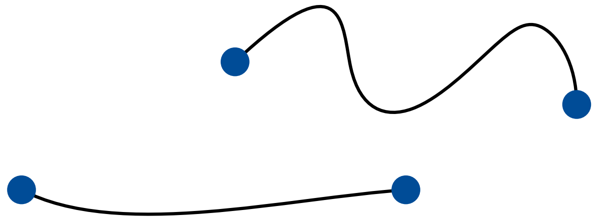

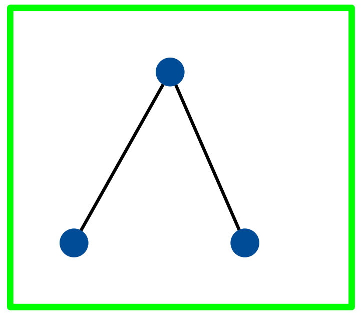

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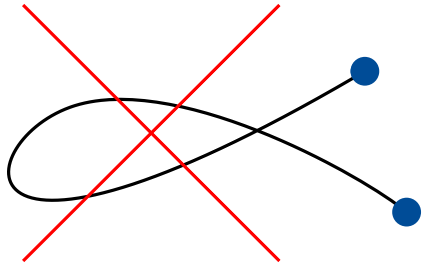

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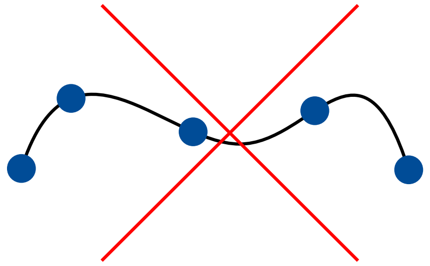

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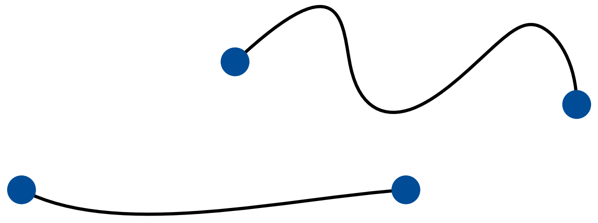

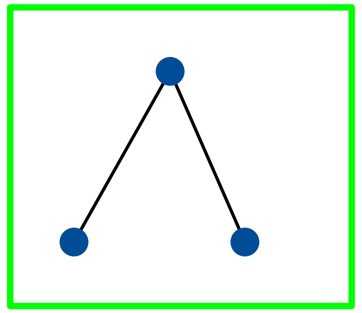

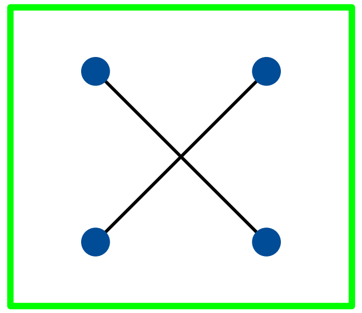

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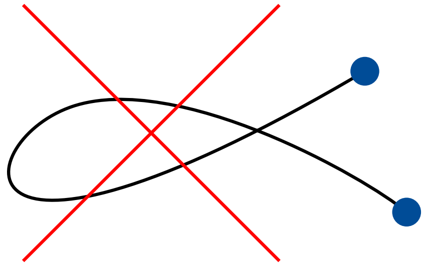

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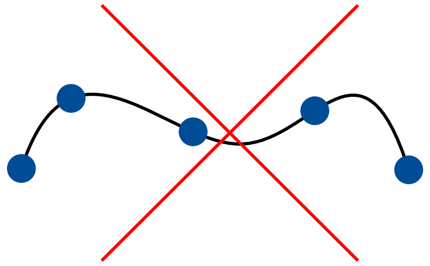

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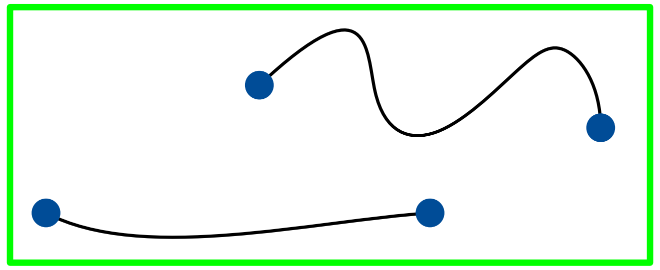

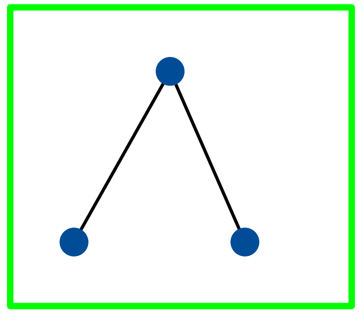

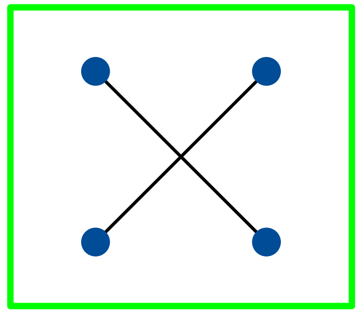

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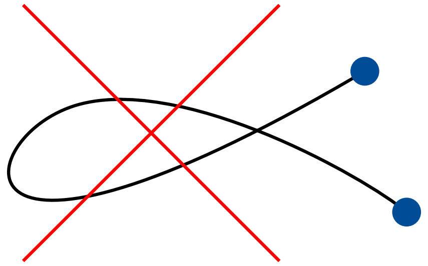

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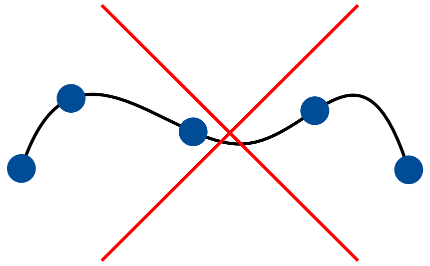

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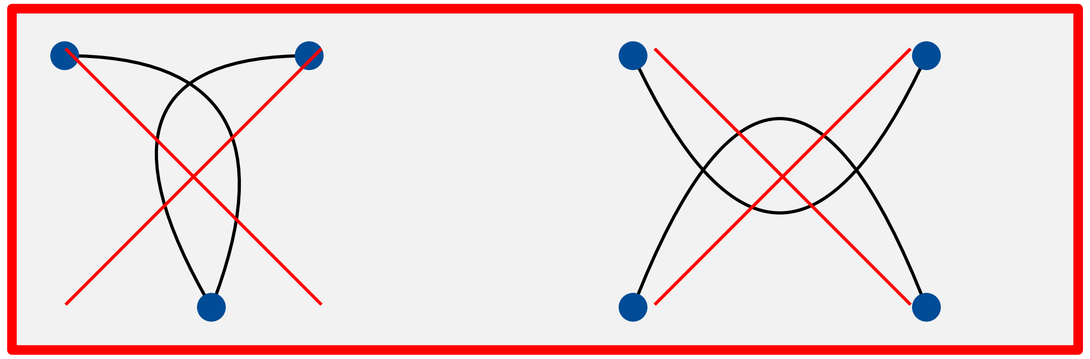

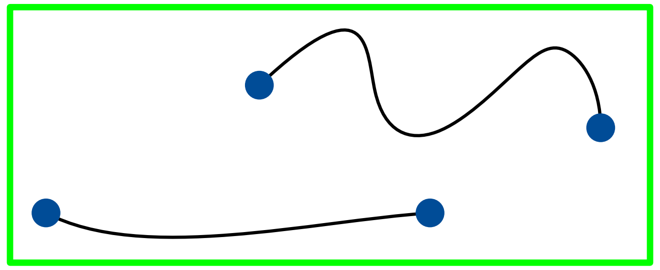

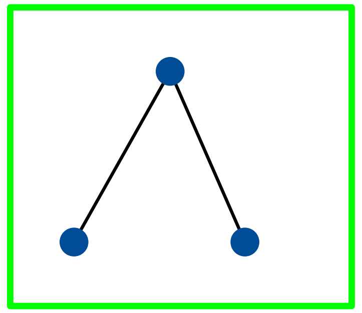

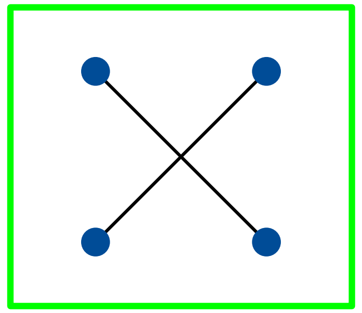

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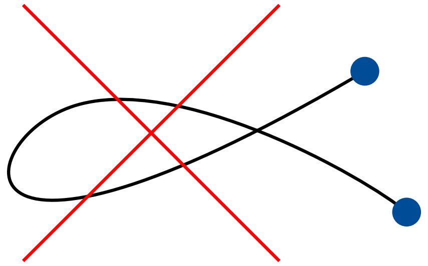

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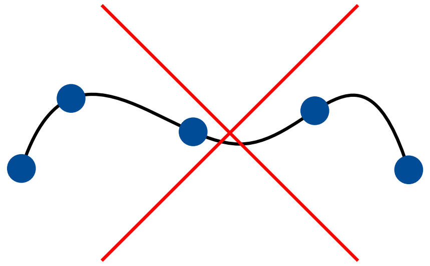

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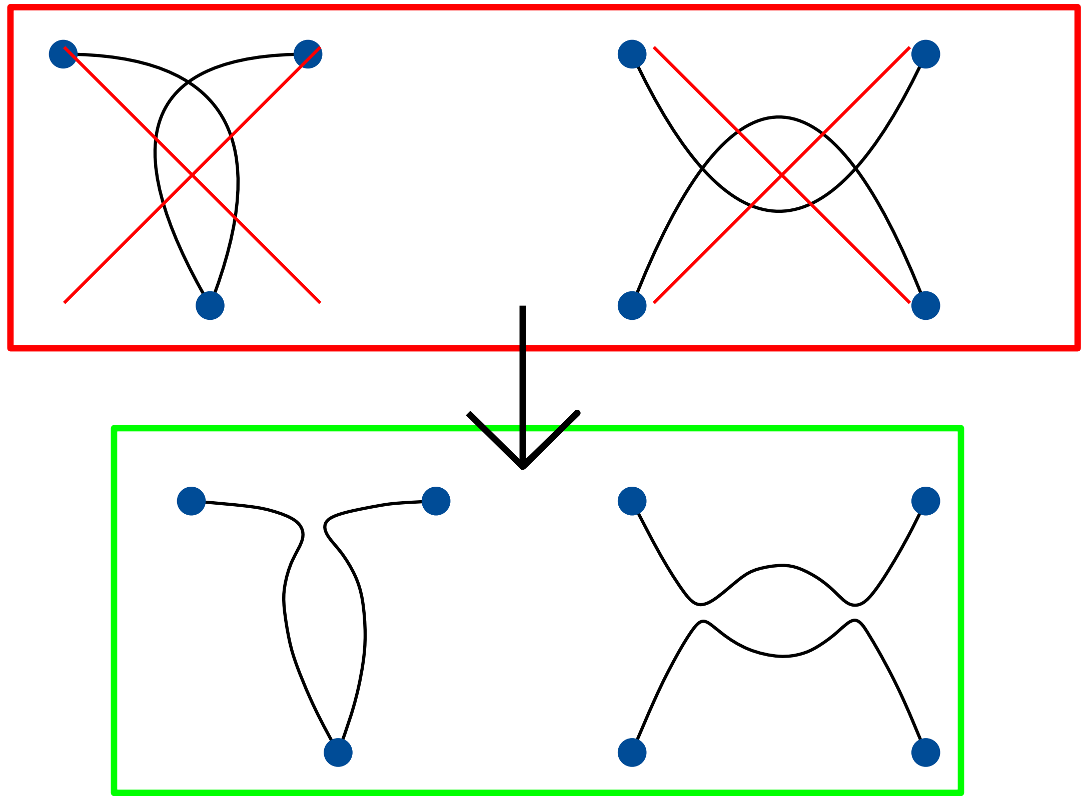

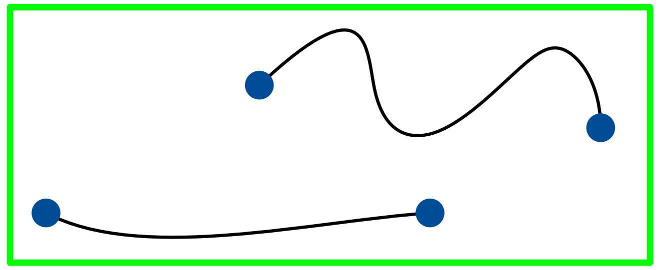

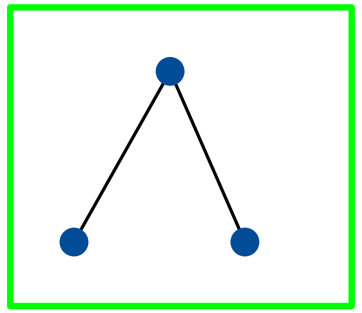

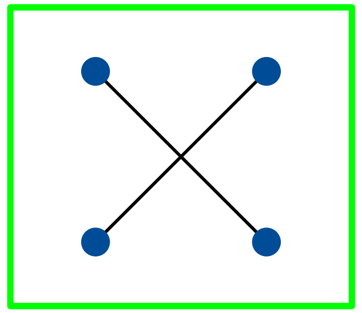

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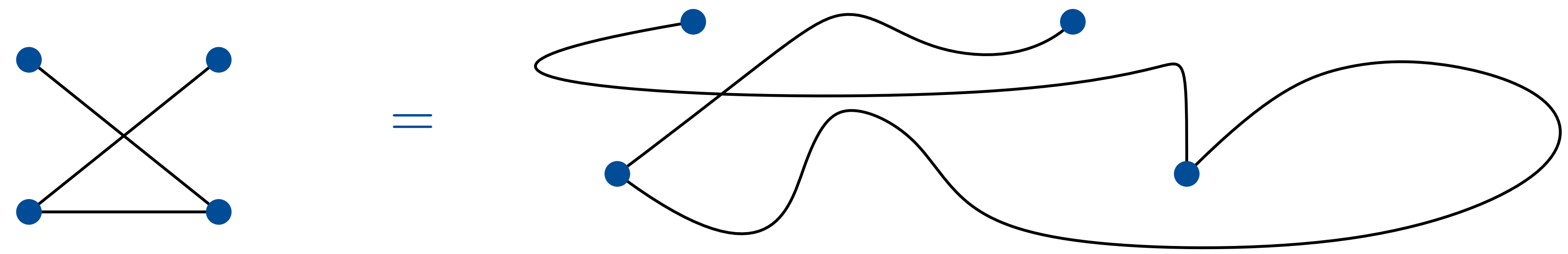
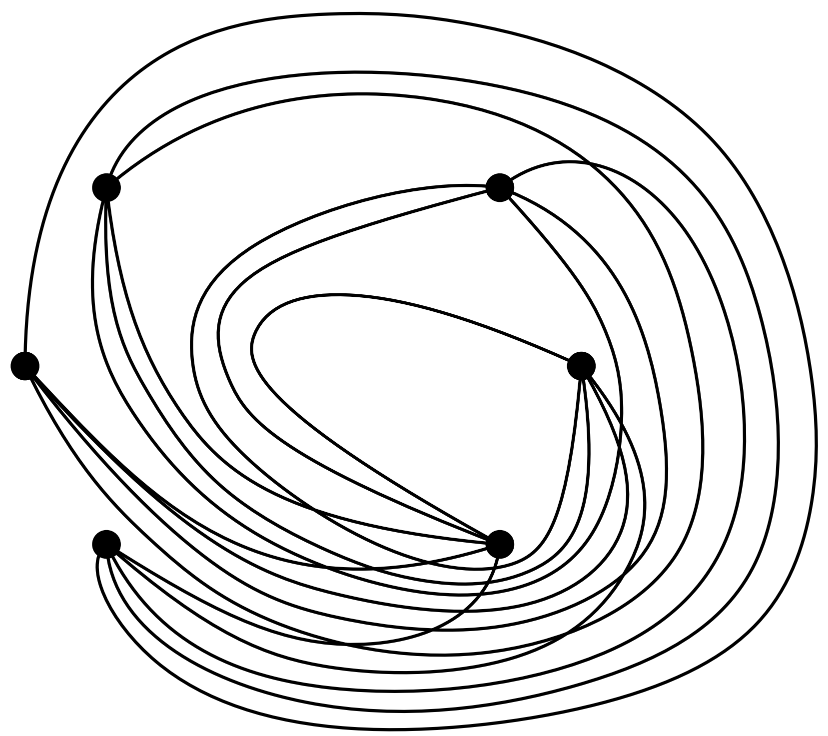
Strongly isomorphic drawings ($\exists$ homeomorphism of the plane transforming one into the other) are considered the same.



Generalized Twisted Drawings

Simple drawing D of K_n is *generalized twisted* if $\exists$ a point O s.t.



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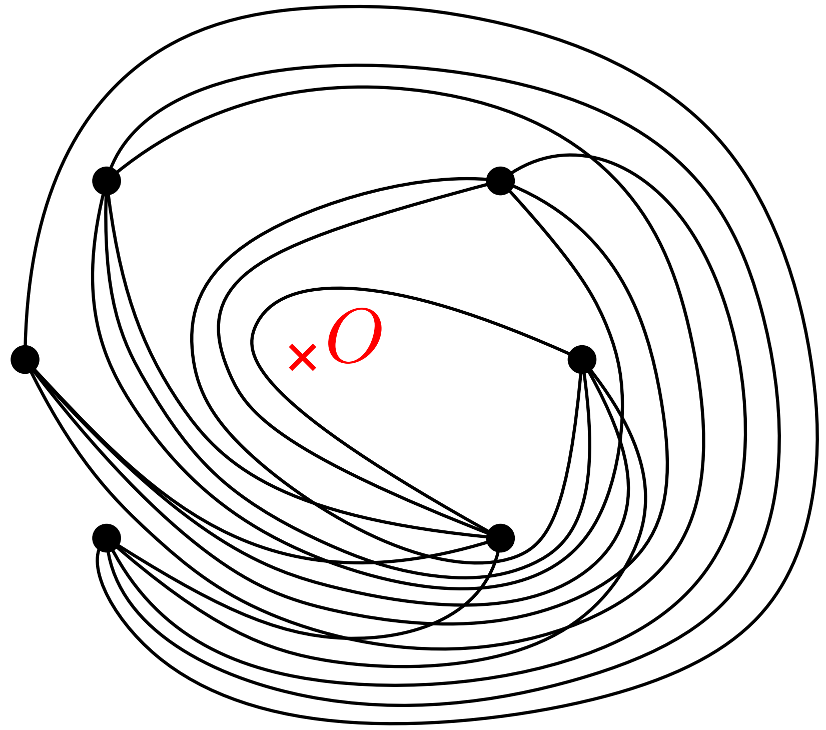

Generalized Twisted Drawings

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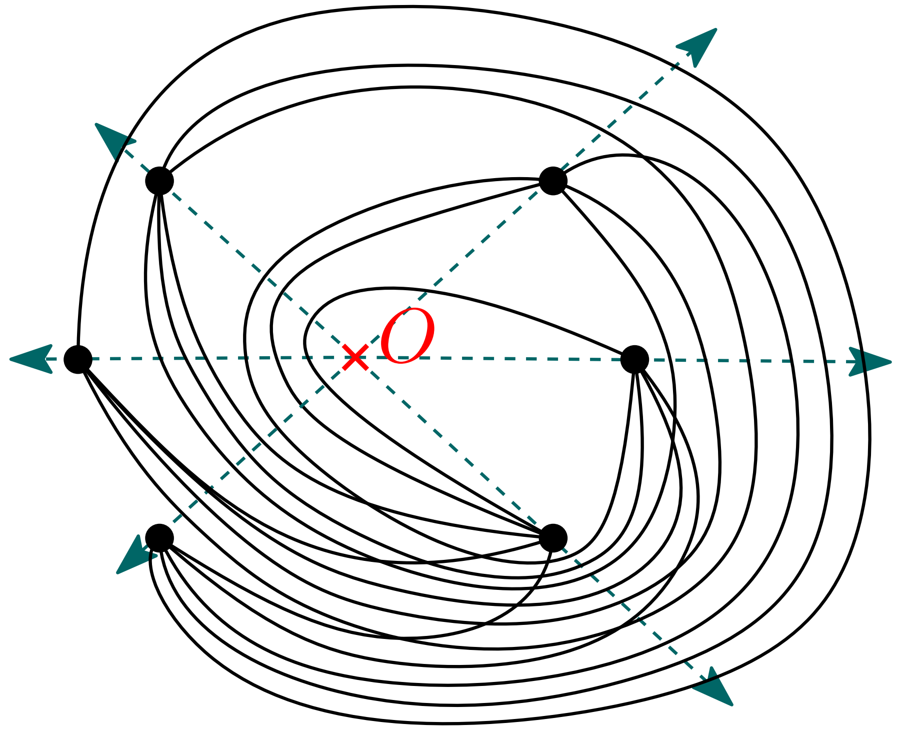
- Any ray emanating from O intersects any edge of D at most once.



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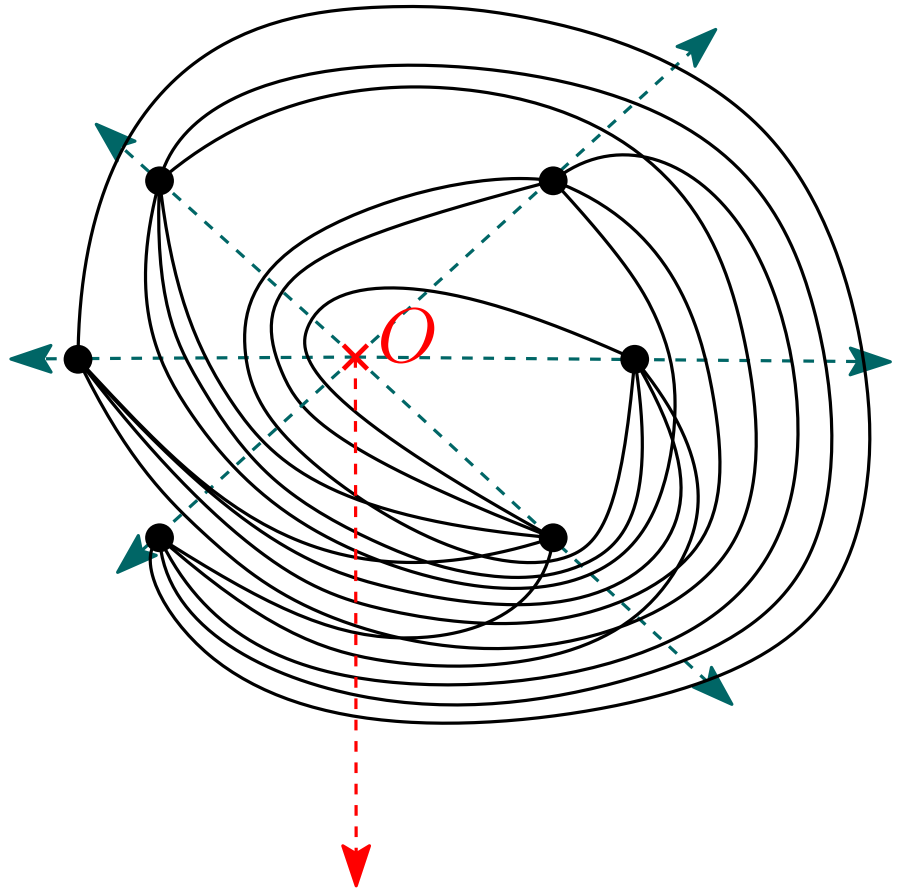

Generalized Twisted Drawings

Simple drawing D of K_n is *generalized twisted* if $\underbrace{\exists \text{ a point } O \text{ s.t.}}$ it is strongly isomorphic to a simple drawing in which

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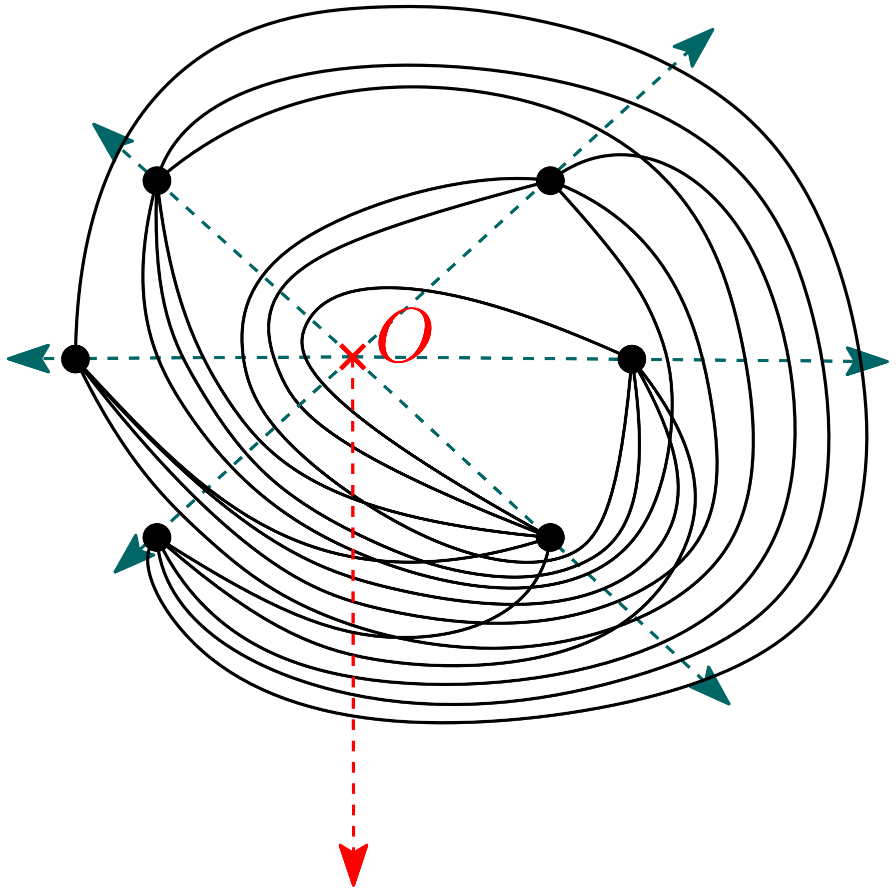

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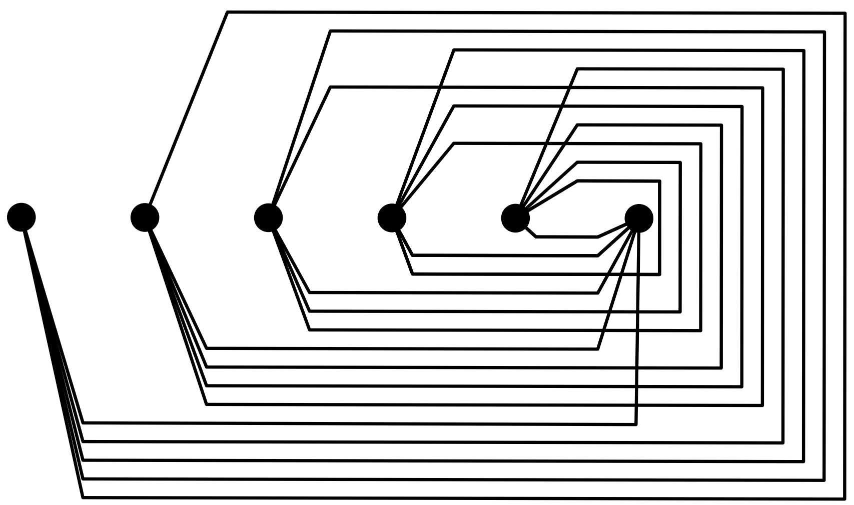

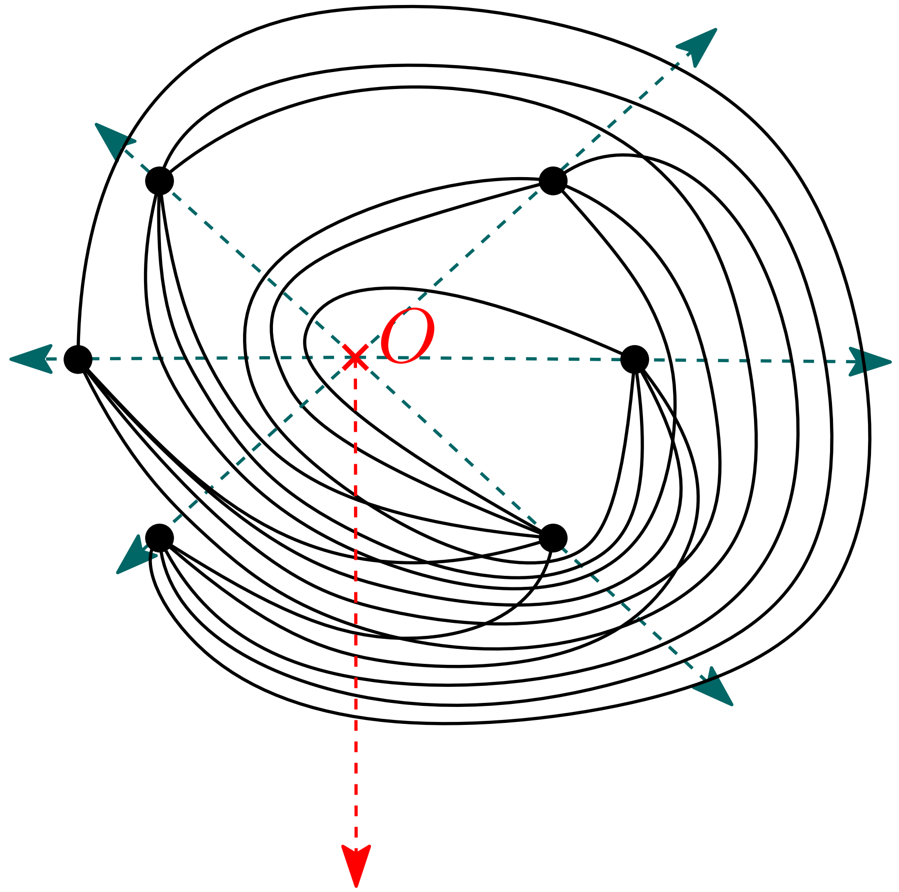

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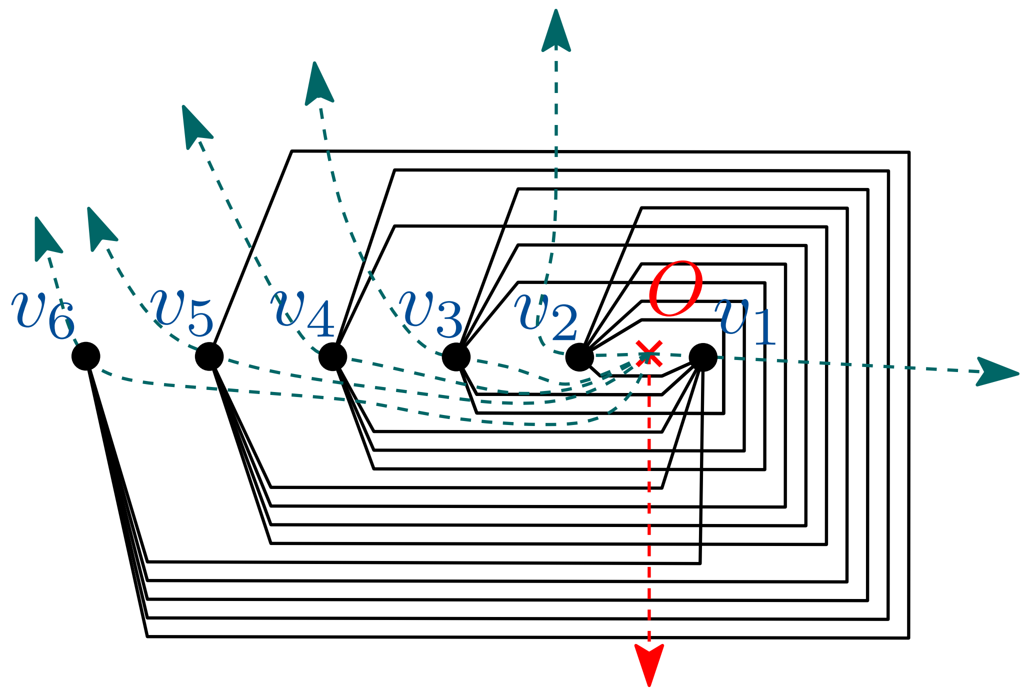

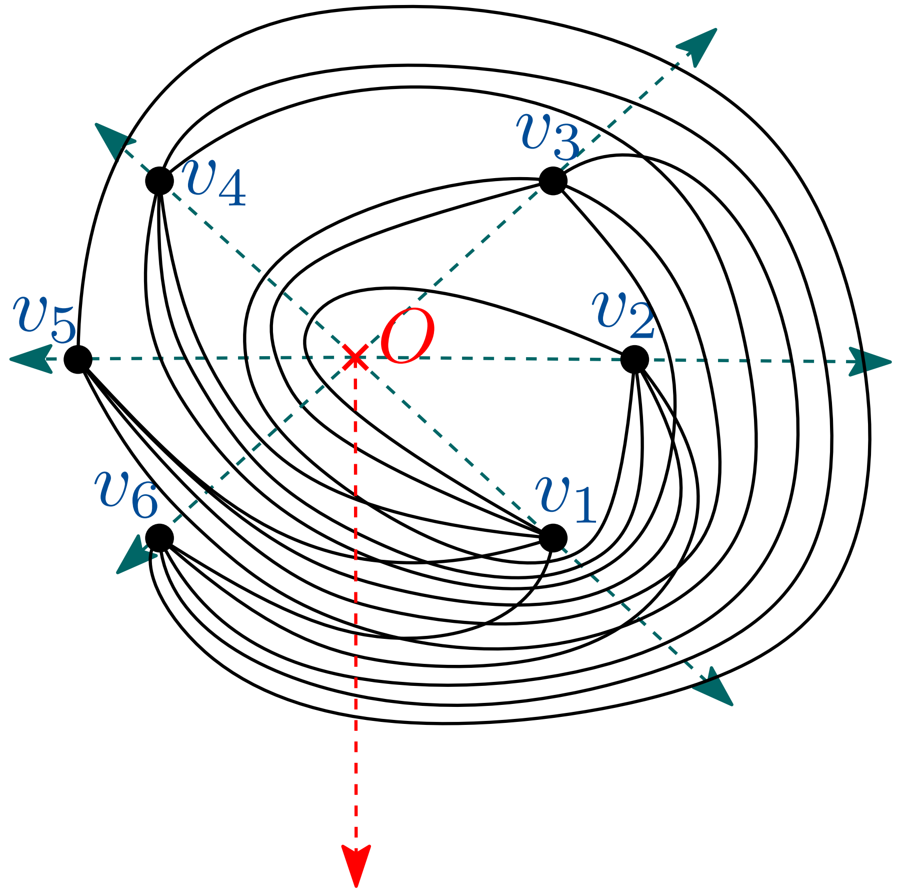

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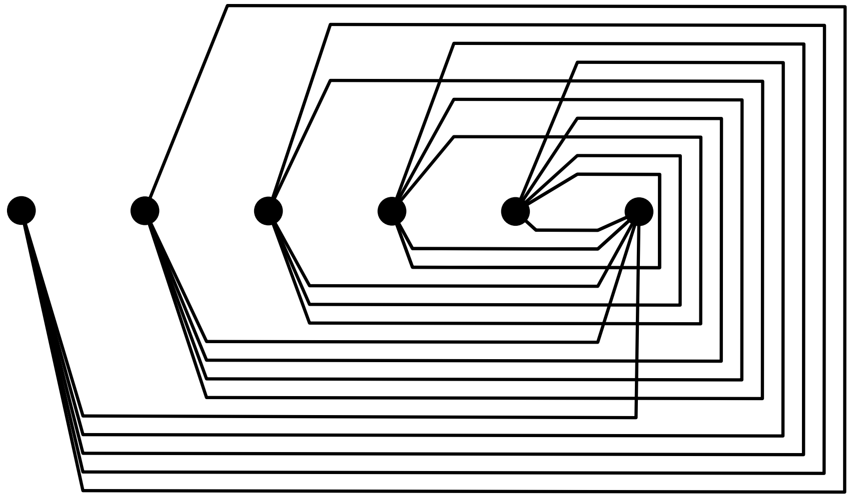

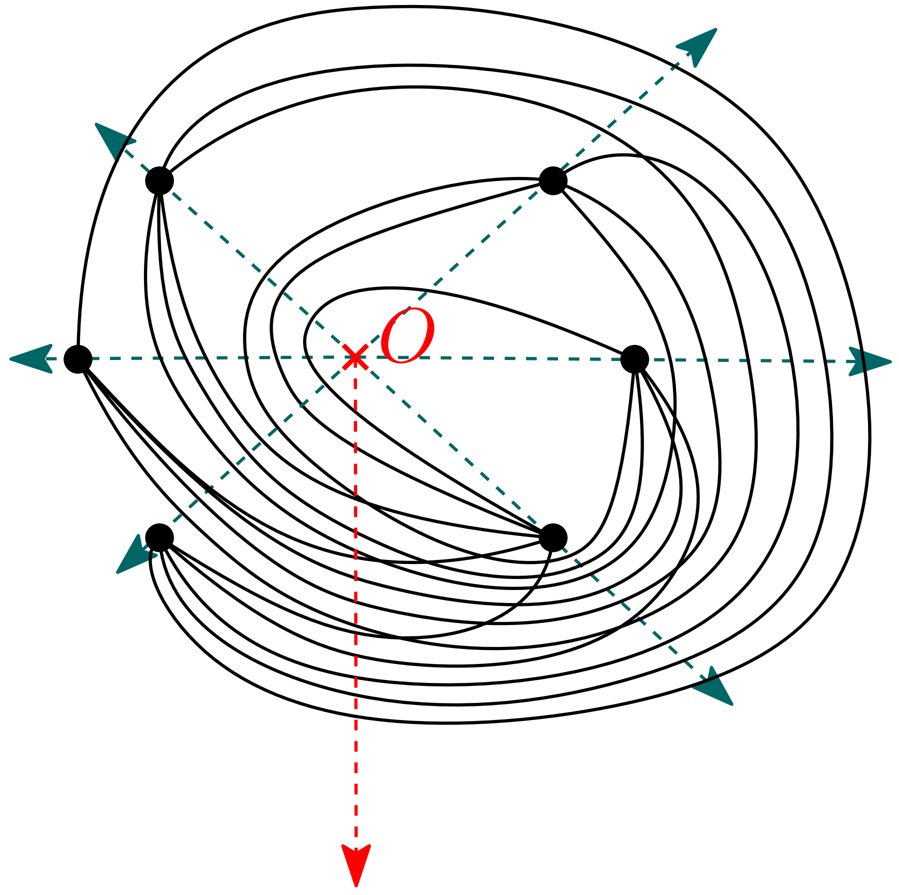

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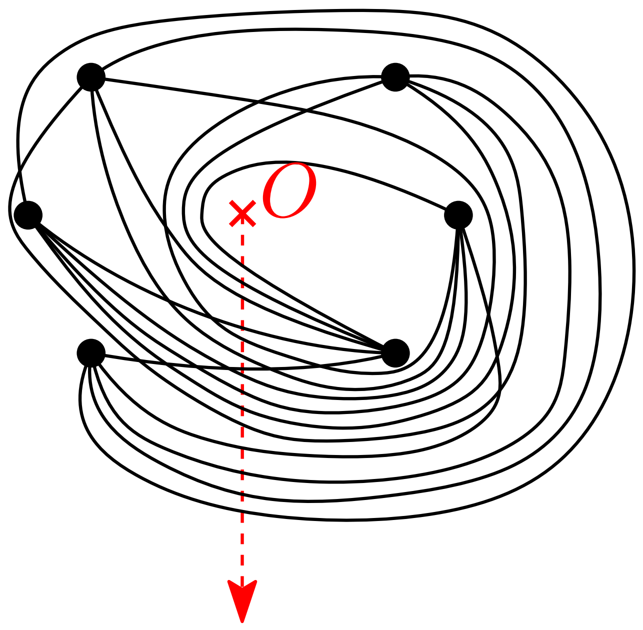

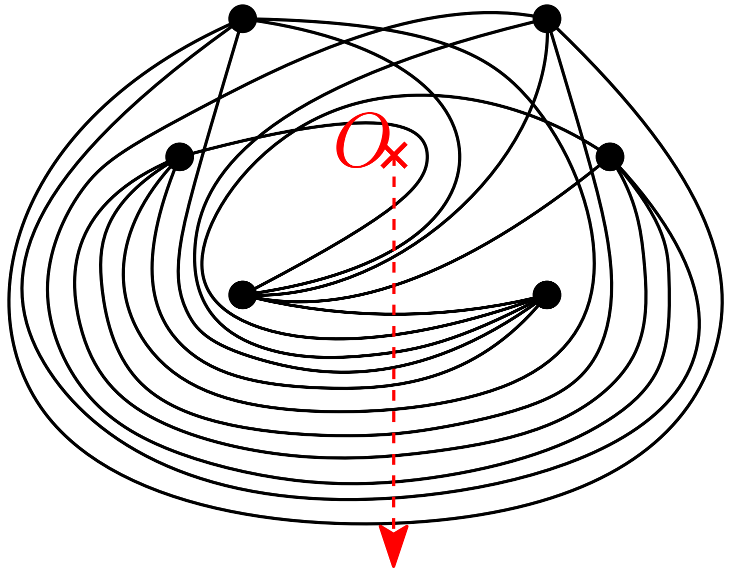

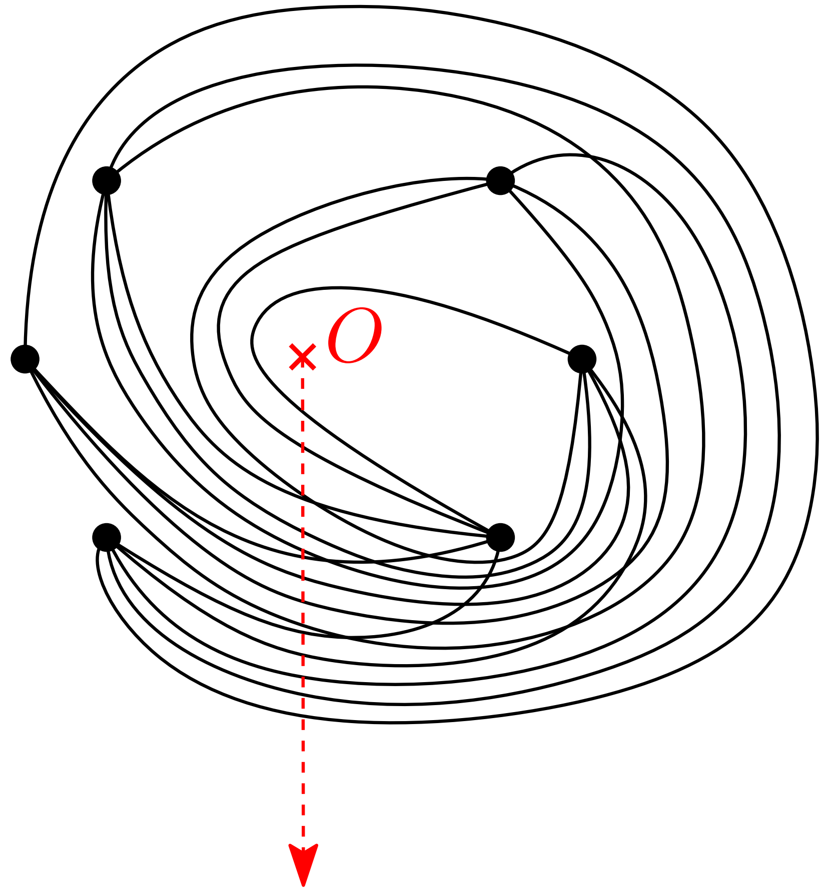

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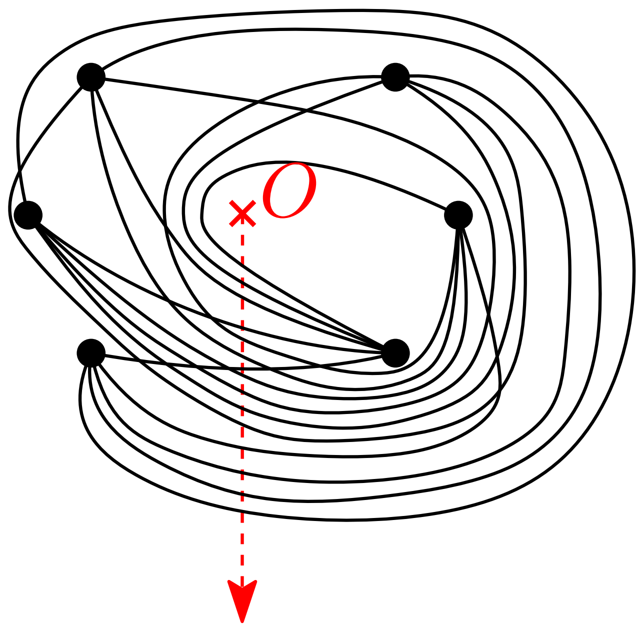
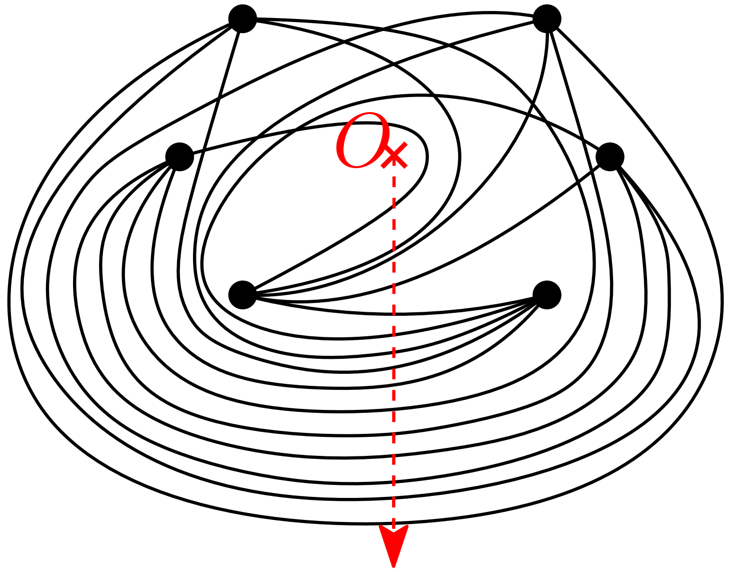
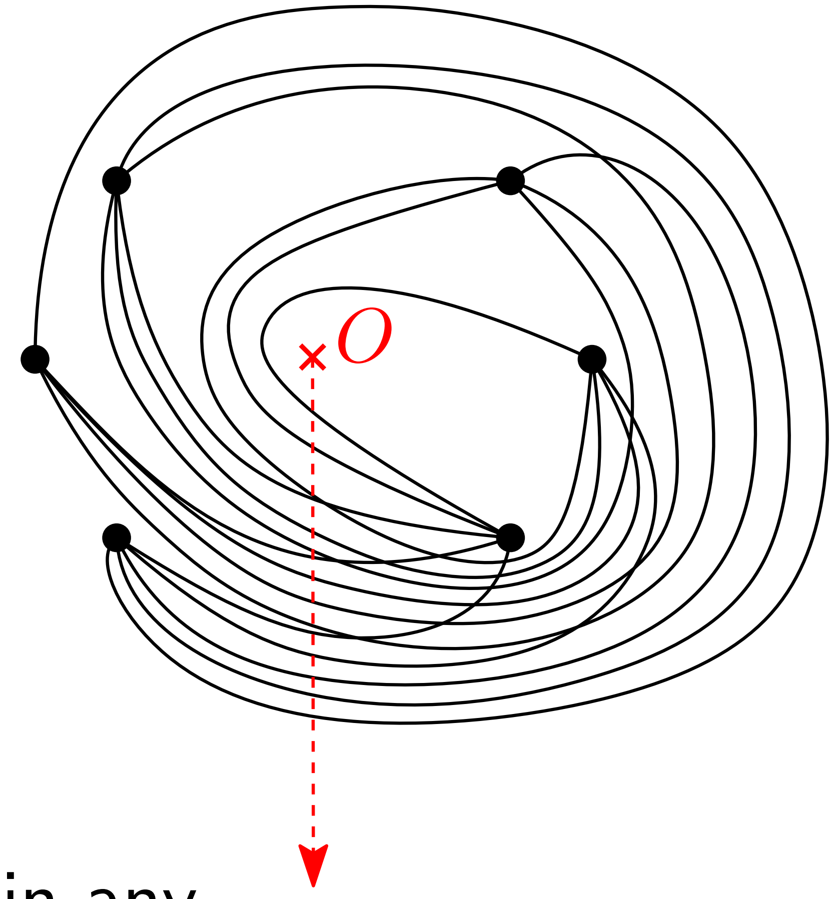


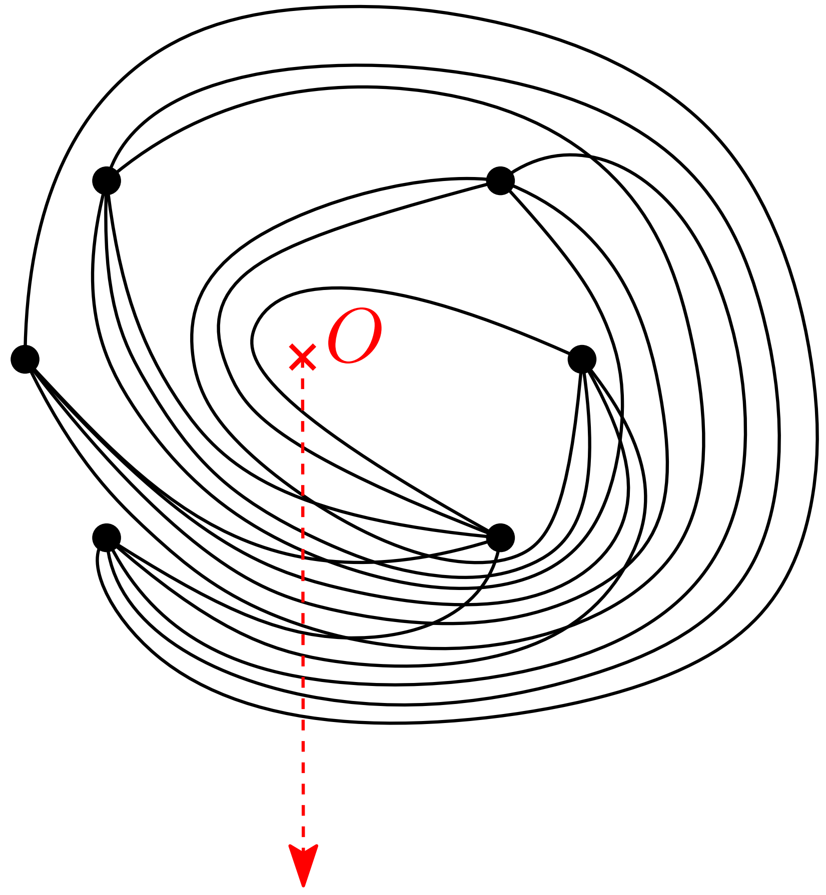




Currently best bounds on size of largest plane matching, cycle and path in any simple drawing of K_n obtained with generalized twisted drawings. [Aichholzer, García, Tejel, Vogtenhuber, W. 2022]

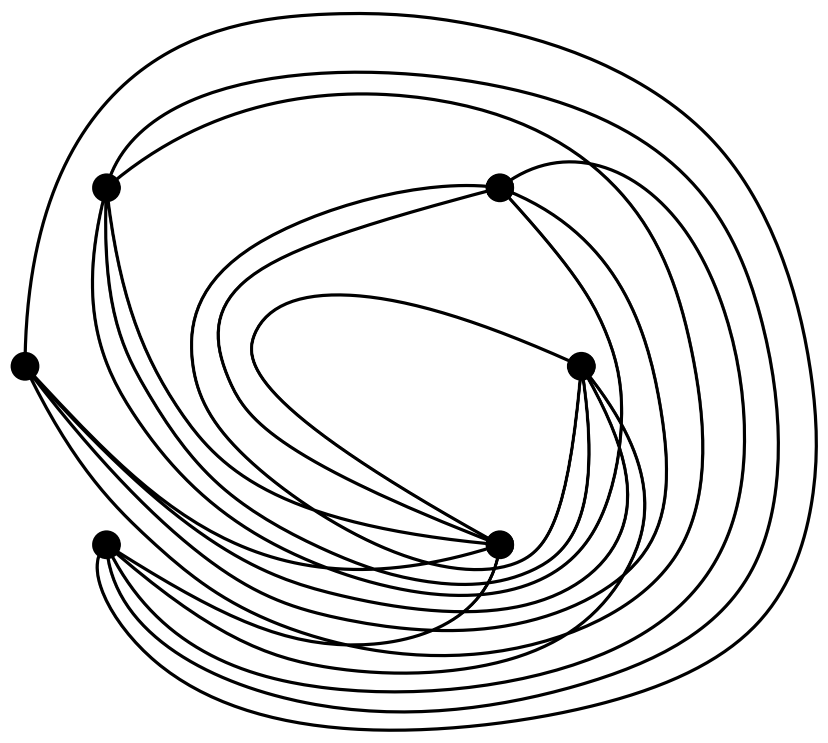
Largest class of drawings where each has exactly $2n-4$ empty triangles. [García, Tejel, Vogtenhuber, W. 2022]

Rotation System



Rotation System

Rotation of vertex in drawing: Cyclical order of its incident edges



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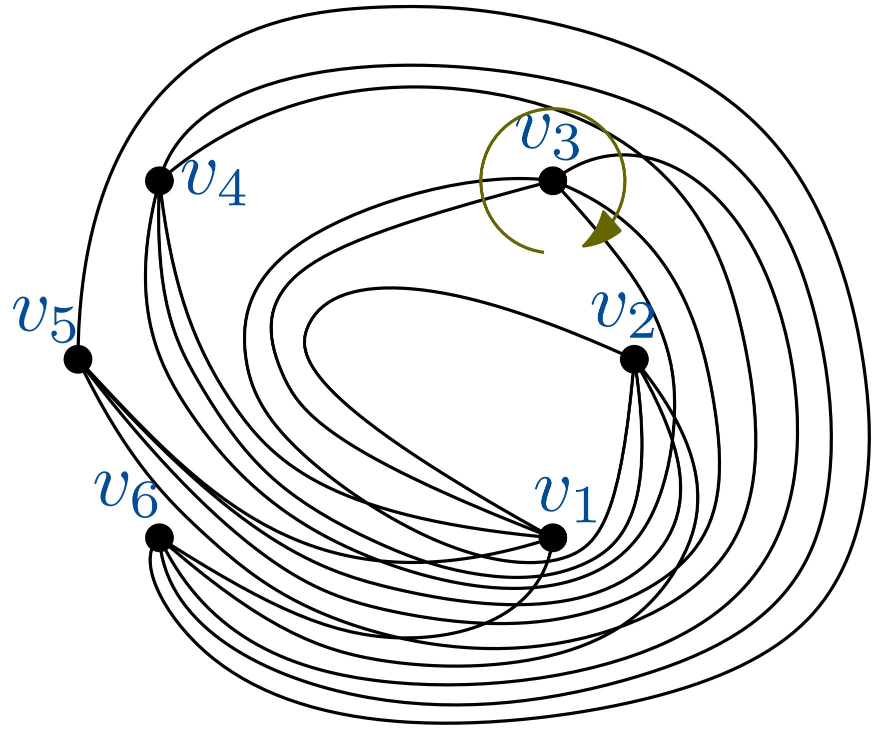
v_3: v_1, v_2, v_6, v_5, v_4



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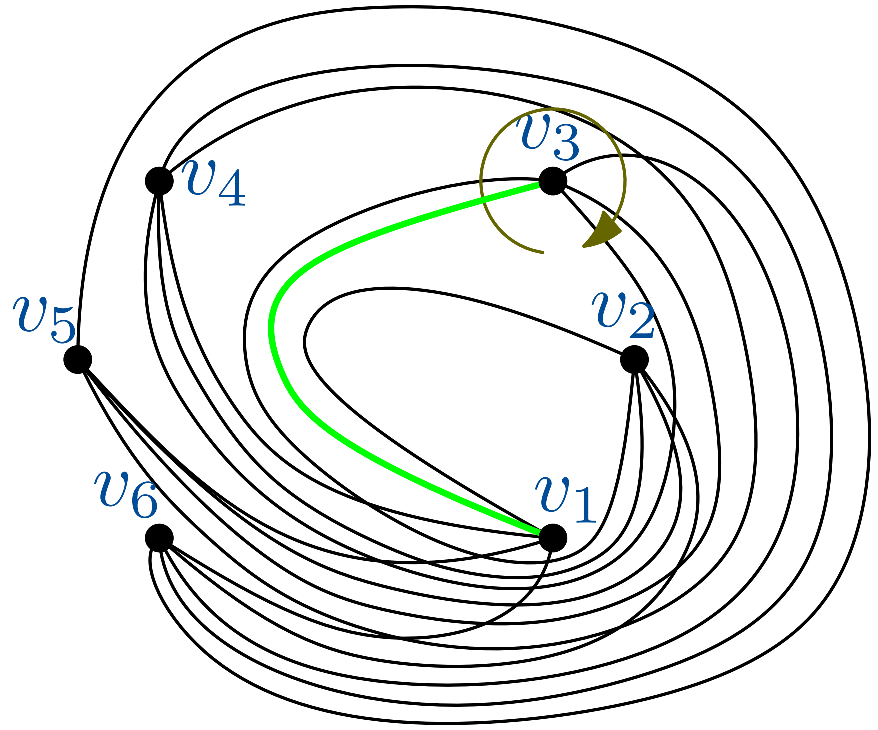
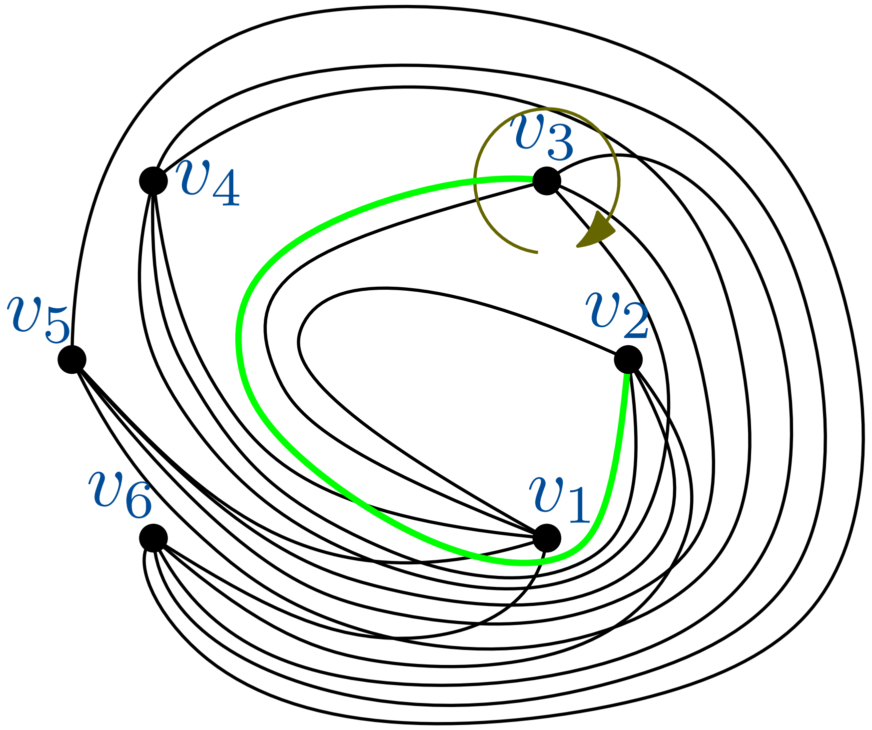


Rotation System

Rotation of vertex in drawing: Cyclical order of its incident edges

v_3: v_1, v_2, v_6, v_5, v_4



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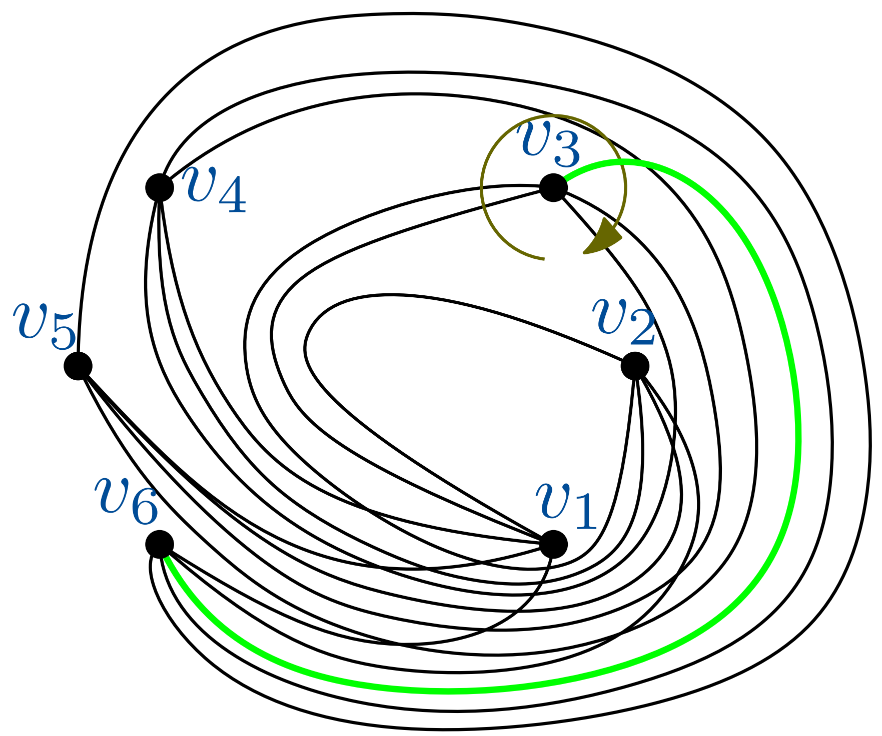

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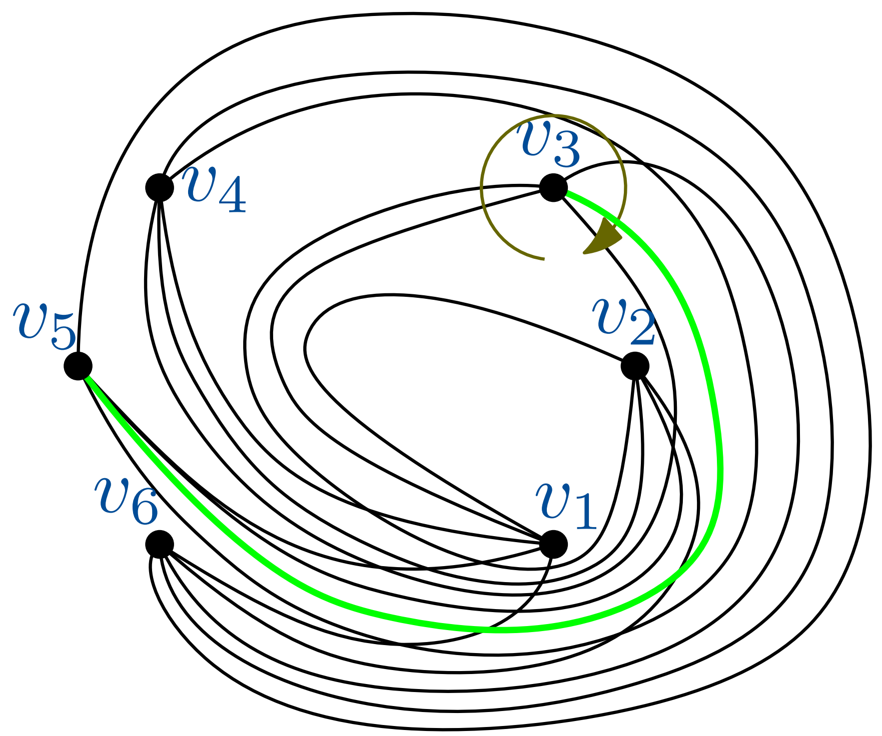

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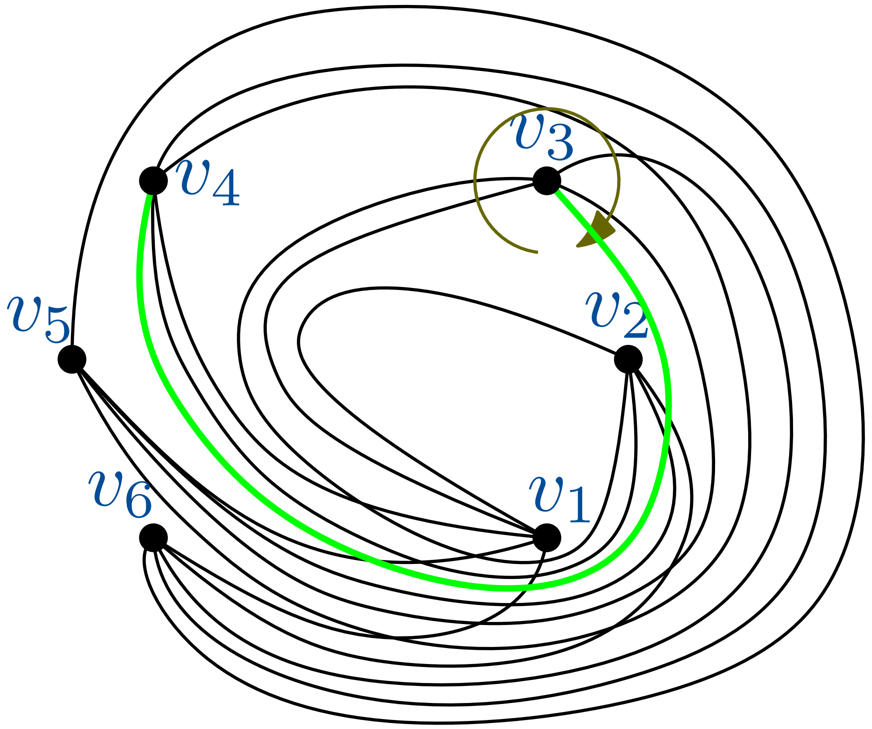

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v_3: v_1, v_2, v_6, v_5, v_4

Rotation system of a drawing: Collection of rotations of all vertices

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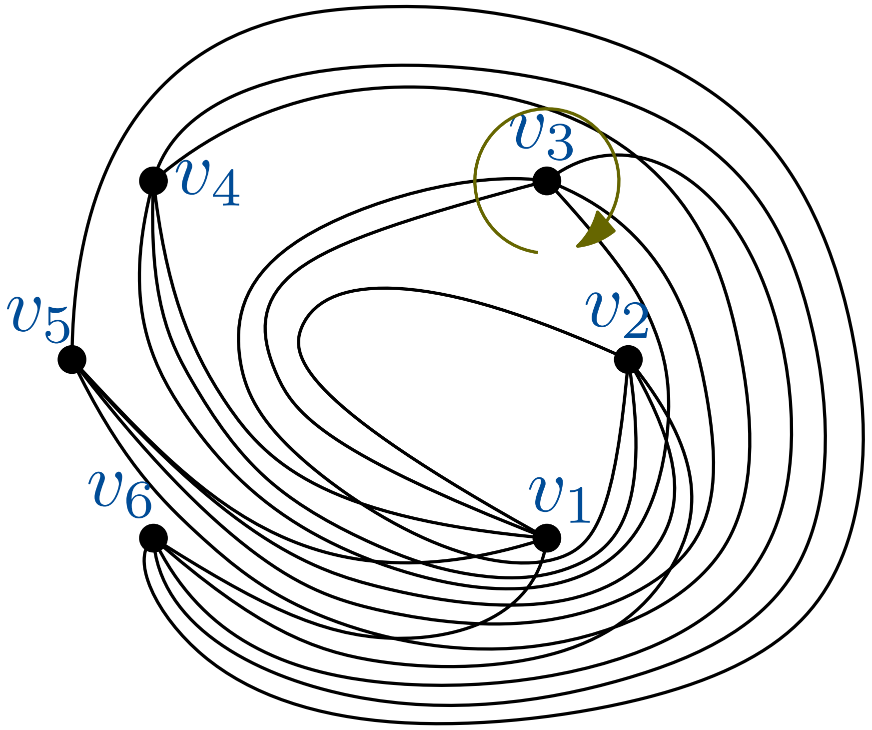

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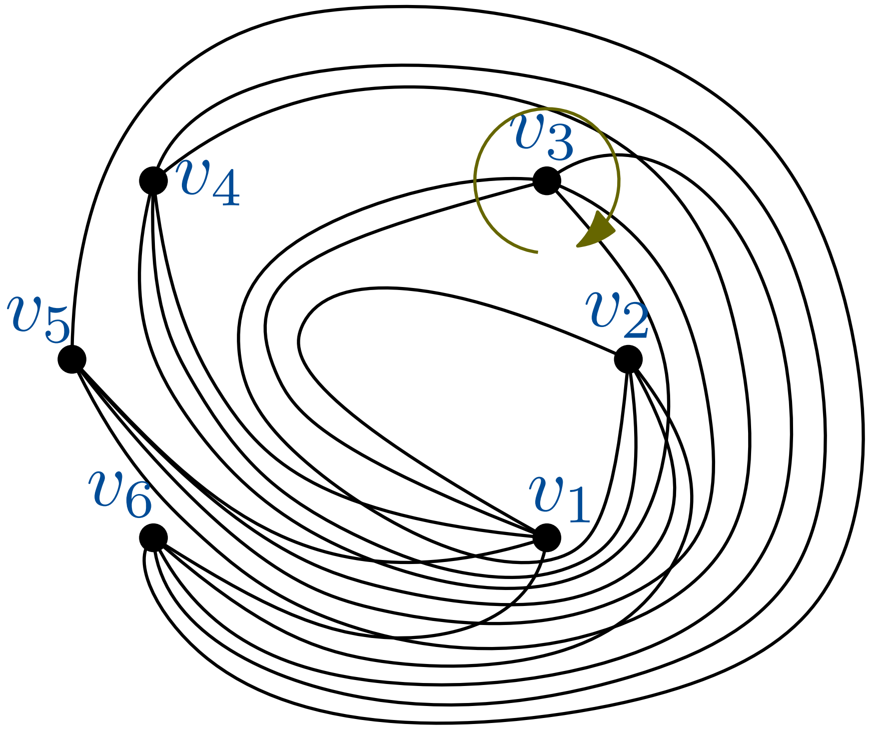

Simple drawings of K_n have same rotation system $\Leftrightarrow$ same crossings [Gioan 2005/2022, Kynčl 2013]

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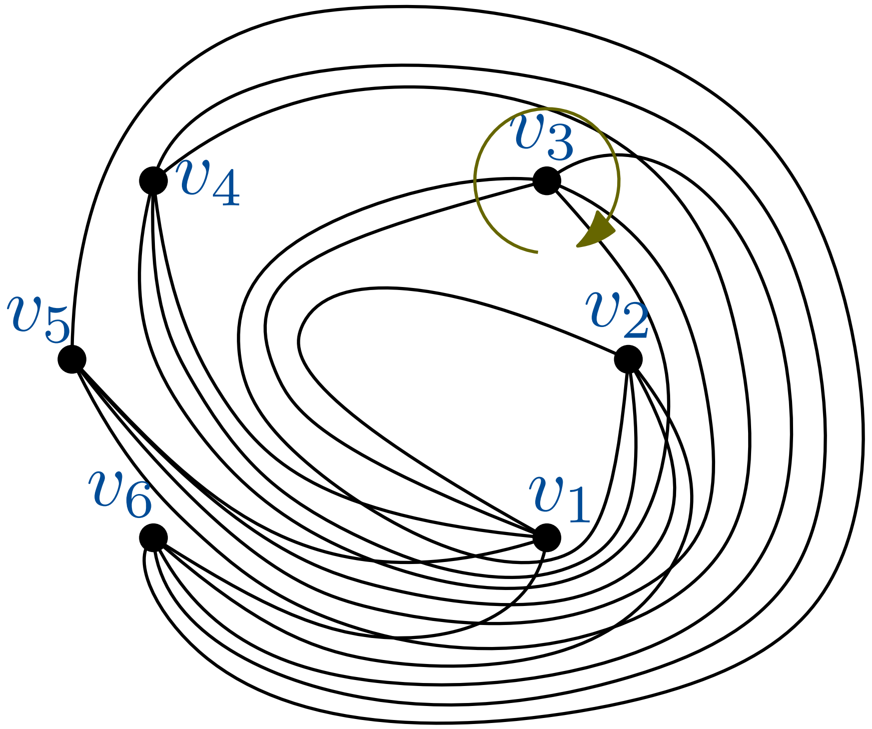

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Several drawing families have been characterized via their rotation system

Rotation System

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Abstract rotation system of a graph: Gives every vertex a cyclic order of its incident edges

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Abstract rotation system of K_n is realizable $\Leftrightarrow$
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Abstract rotation system is generalized twisted iff
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Characterizing Twistedness Using 5-Tuples

An abstract rotation system of K_n with $n \geq 7$ is generalized twisted
$\Leftrightarrow$ every subrotation system induced by 5 vertices is.

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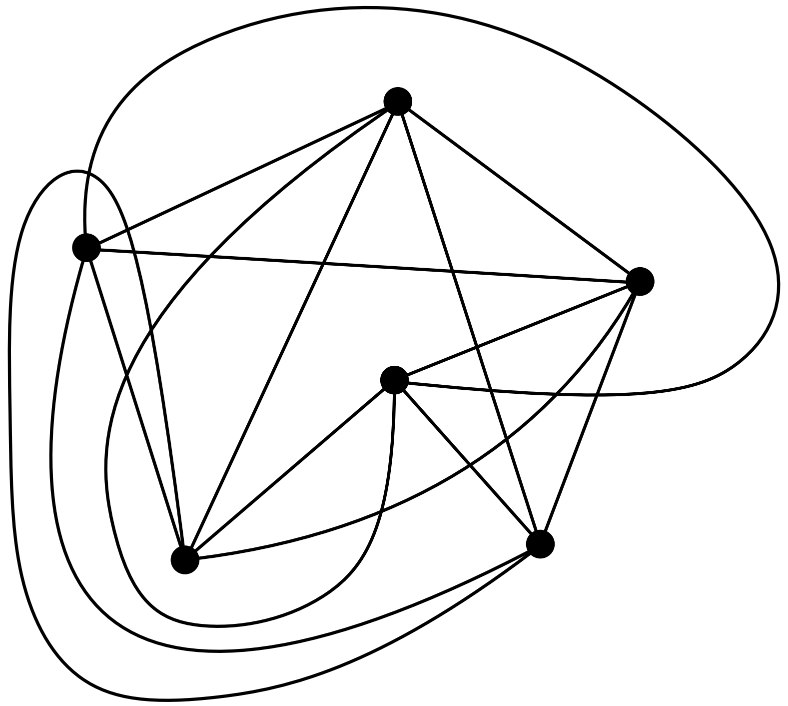
An abstract rotation system of K_n with $n \geq 7$ is generalized twisted $\Leftrightarrow$ every subrotation system induced by 5 vertices is.

For $n = 6$ this is not the case.



Characterizing Twistedness Using a Pair of Vertices and a Bipartition

A realizable rotation system of K_n with $n \geq 3$ is generalized twisted
$\Leftrightarrow$ there exist two vertices v_1 and v_2 and a bipartition $A \cup B$ of the remaining vertices
(A or B can be empty) such that:

- $\forall$ a, a' in A, the edge aa' crosses v_1v_2.
- $\forall$ b, b' in B, the edge bb' crosses v_1v_2.
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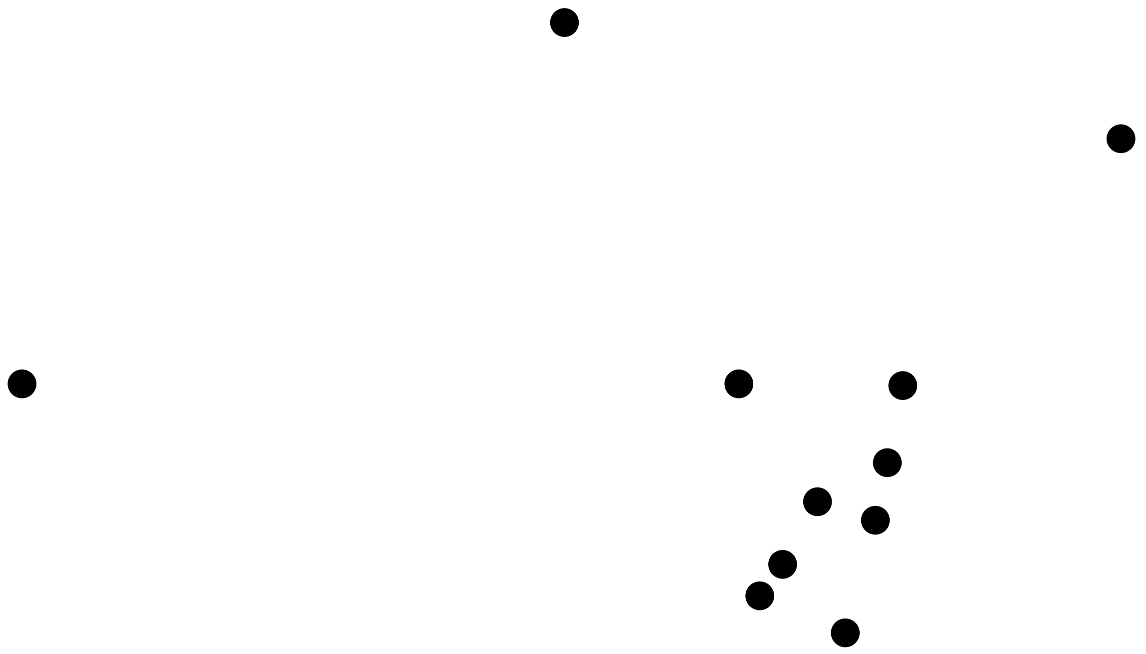

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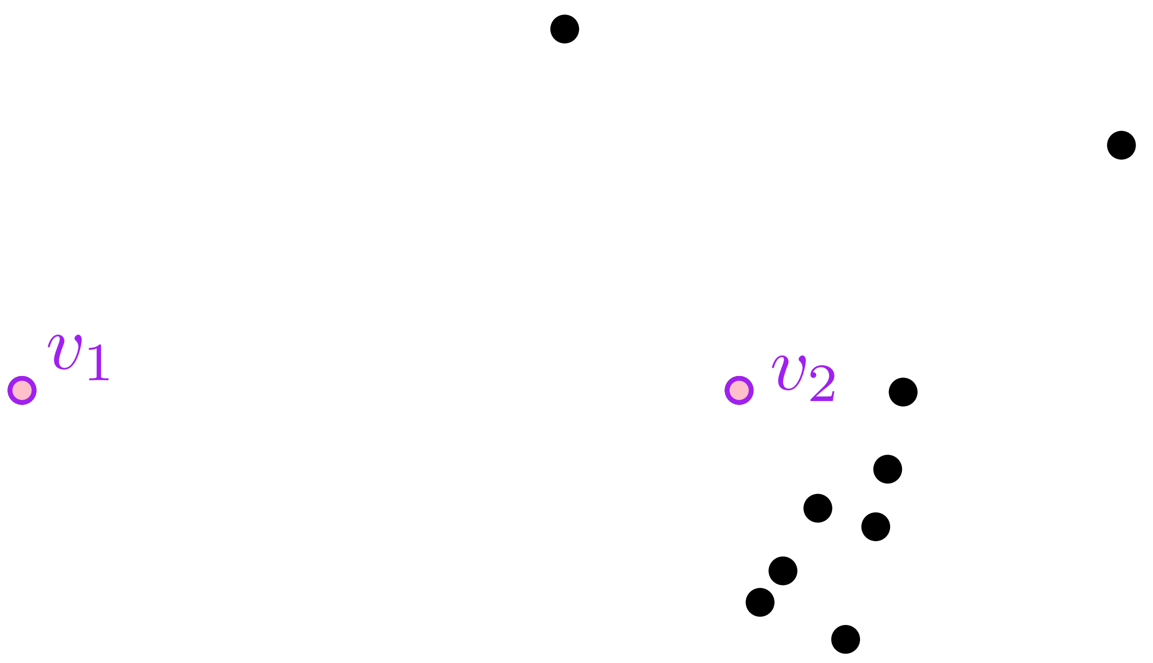

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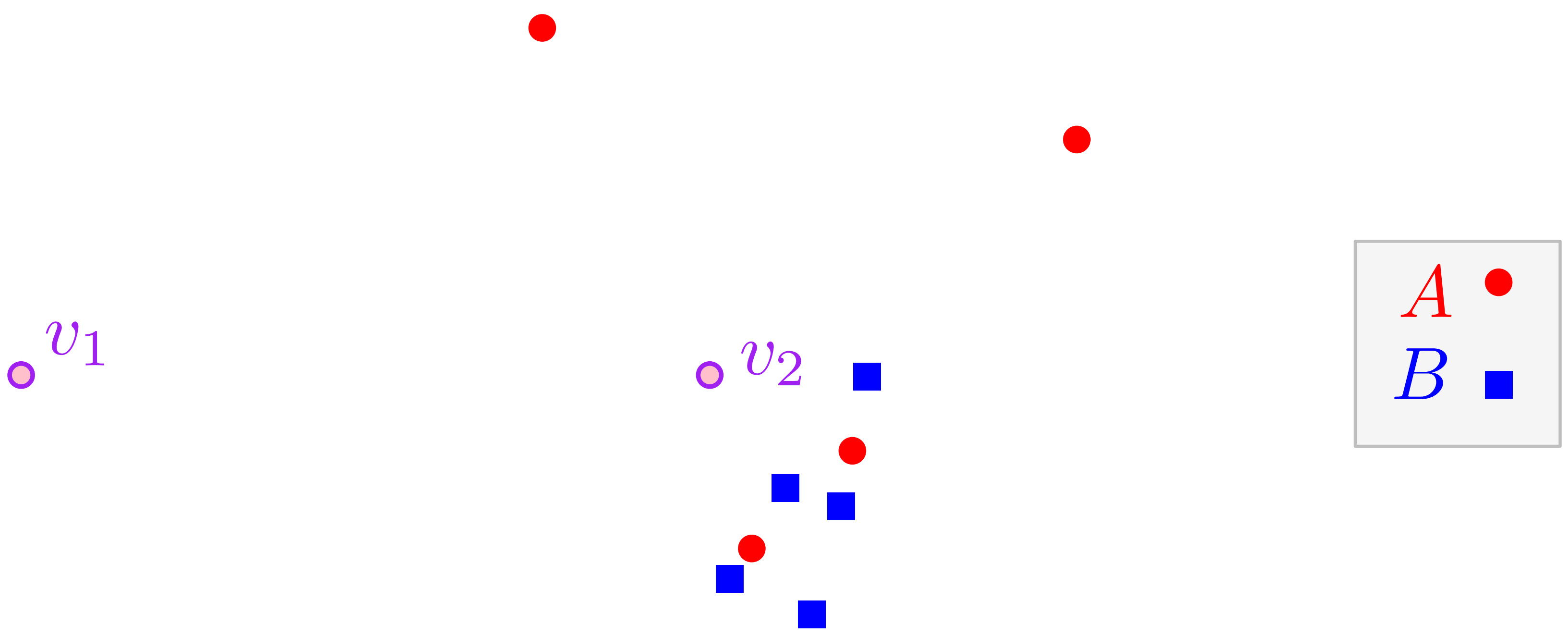

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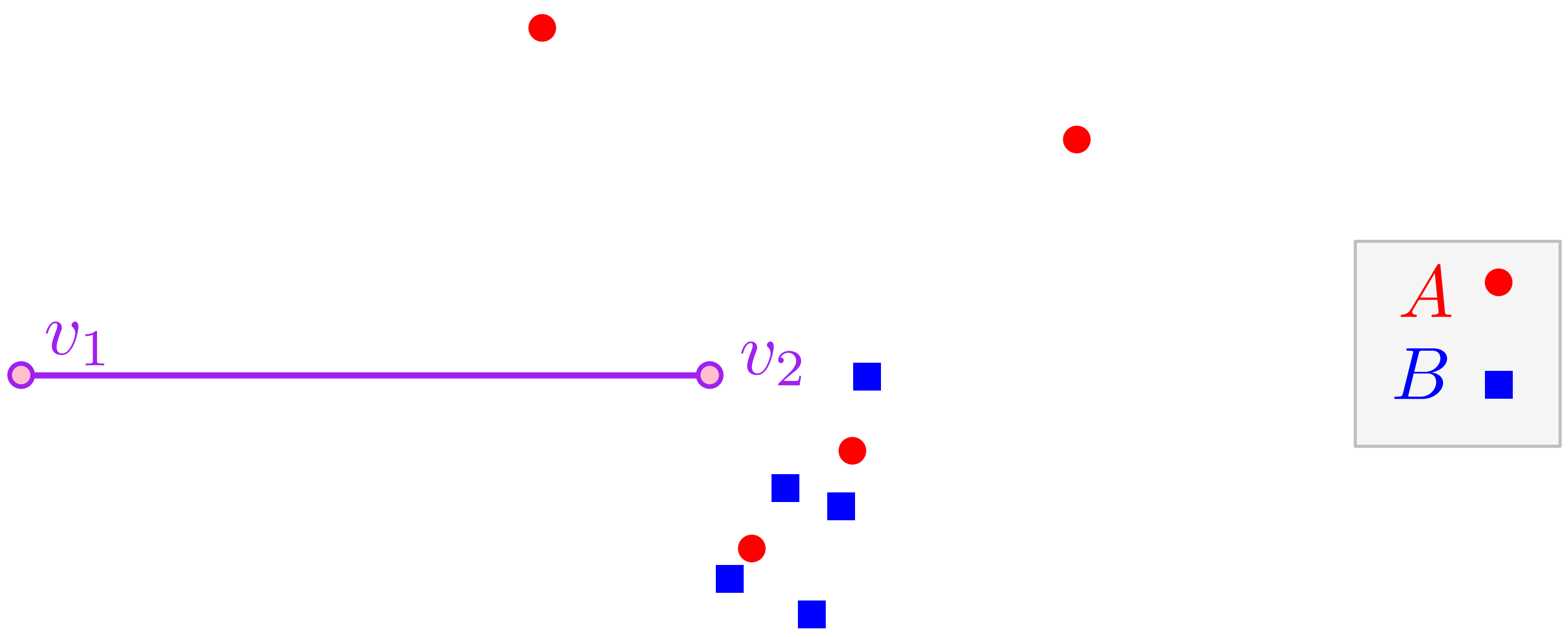

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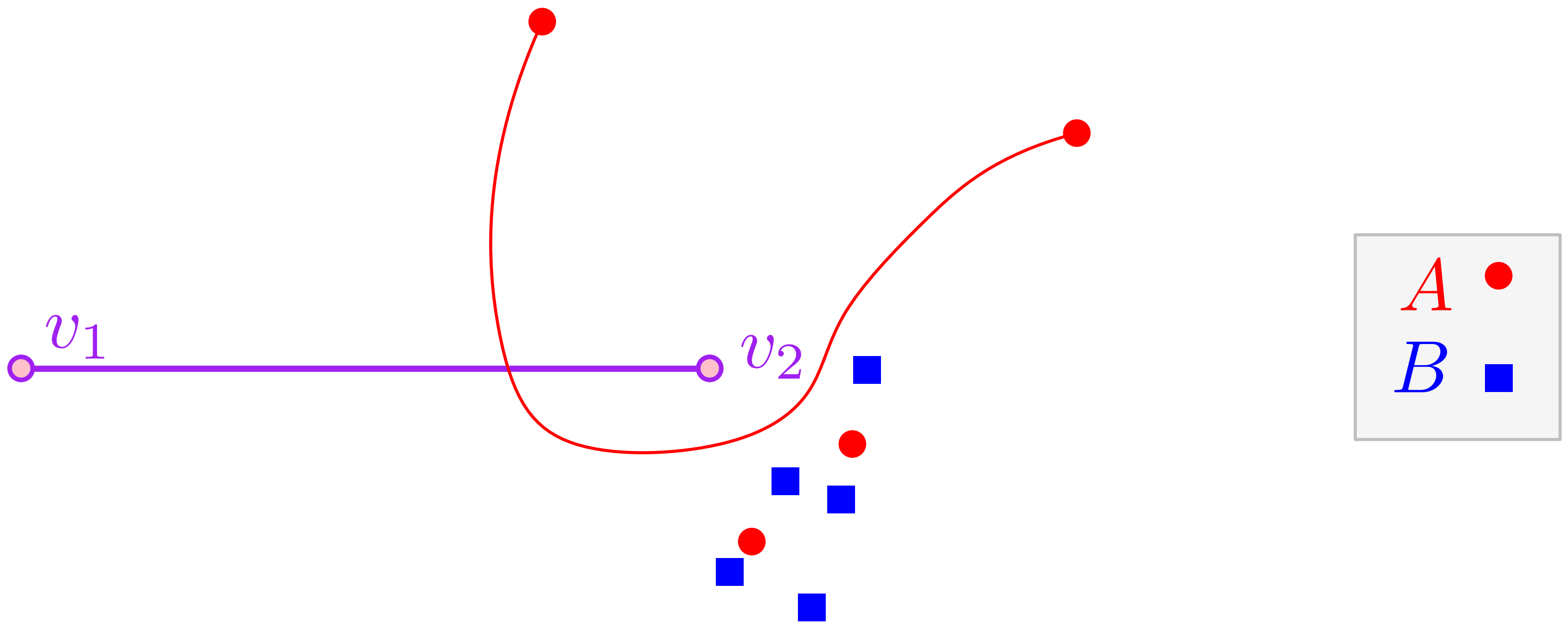

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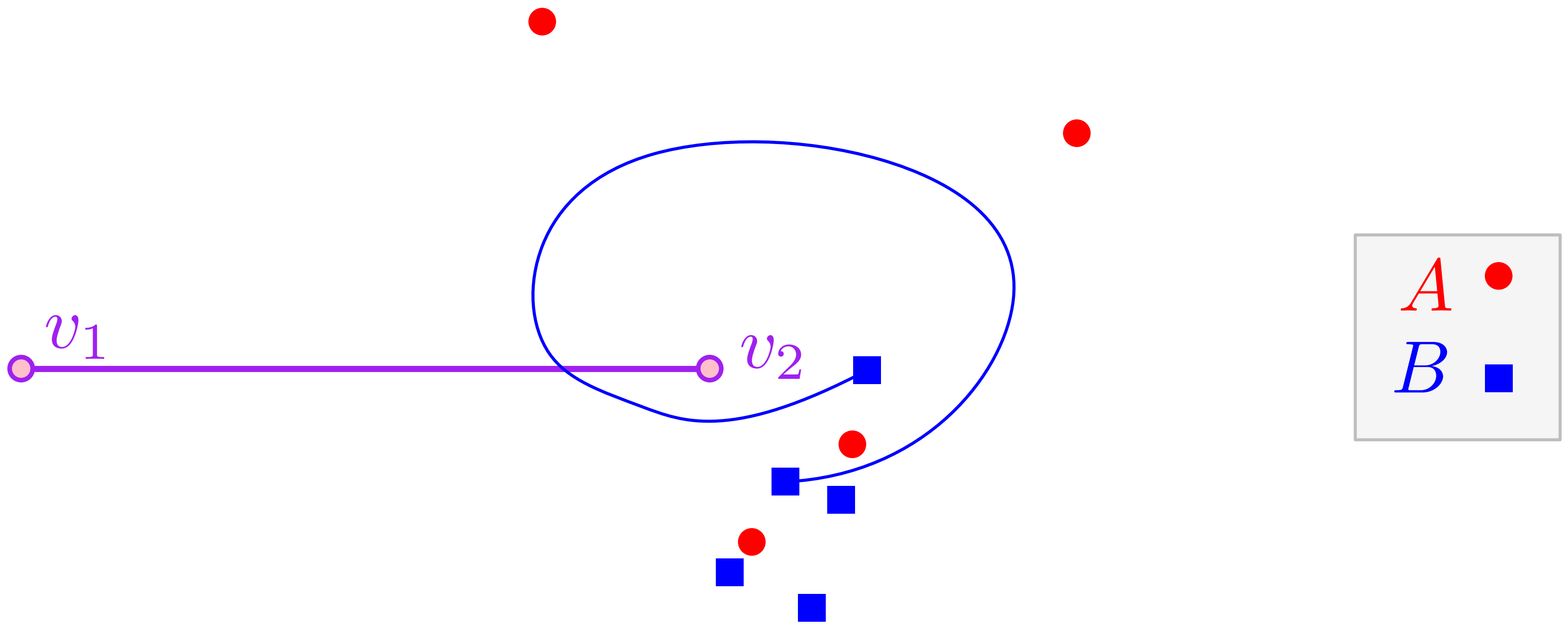

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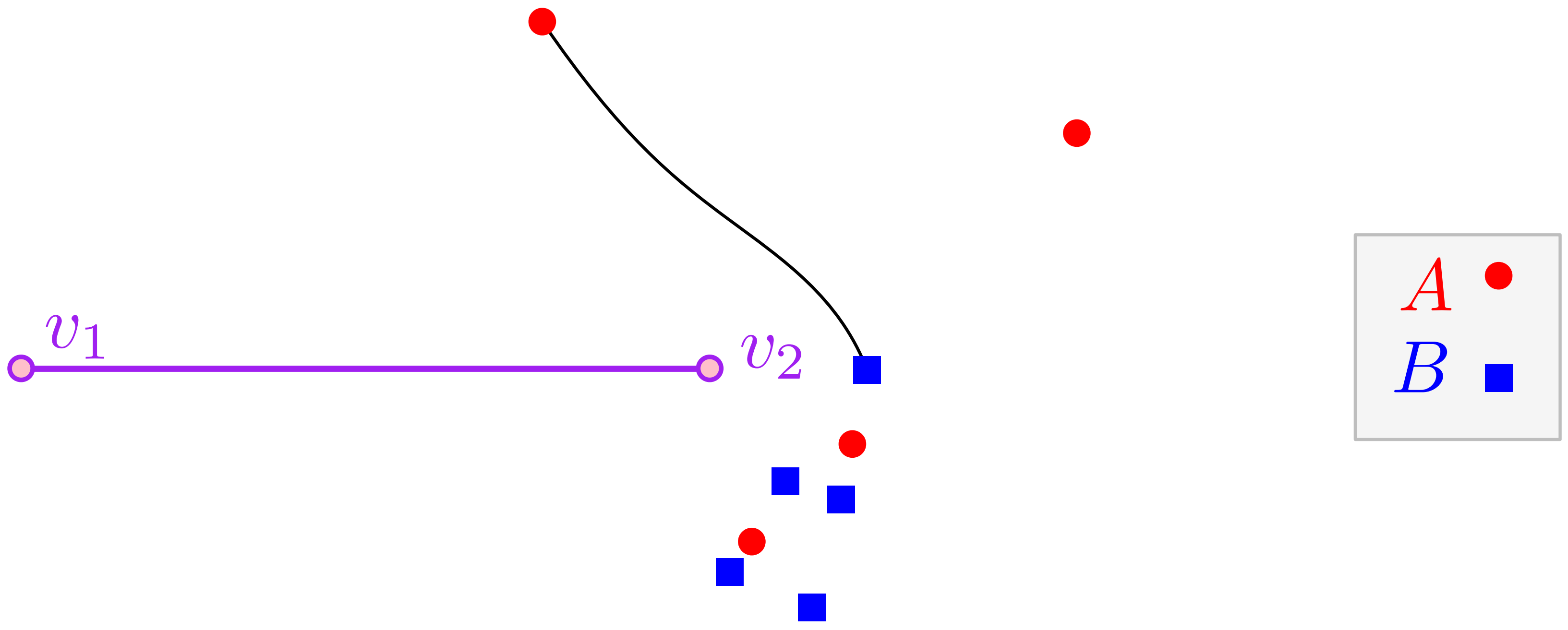

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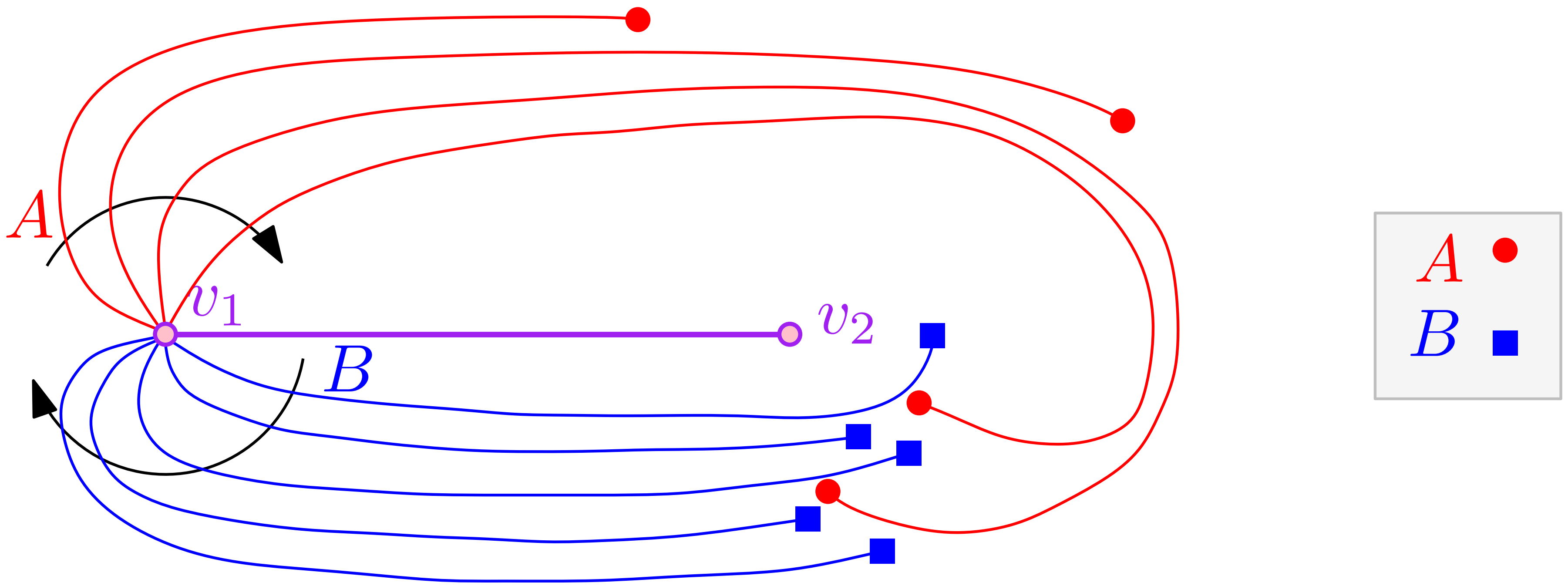

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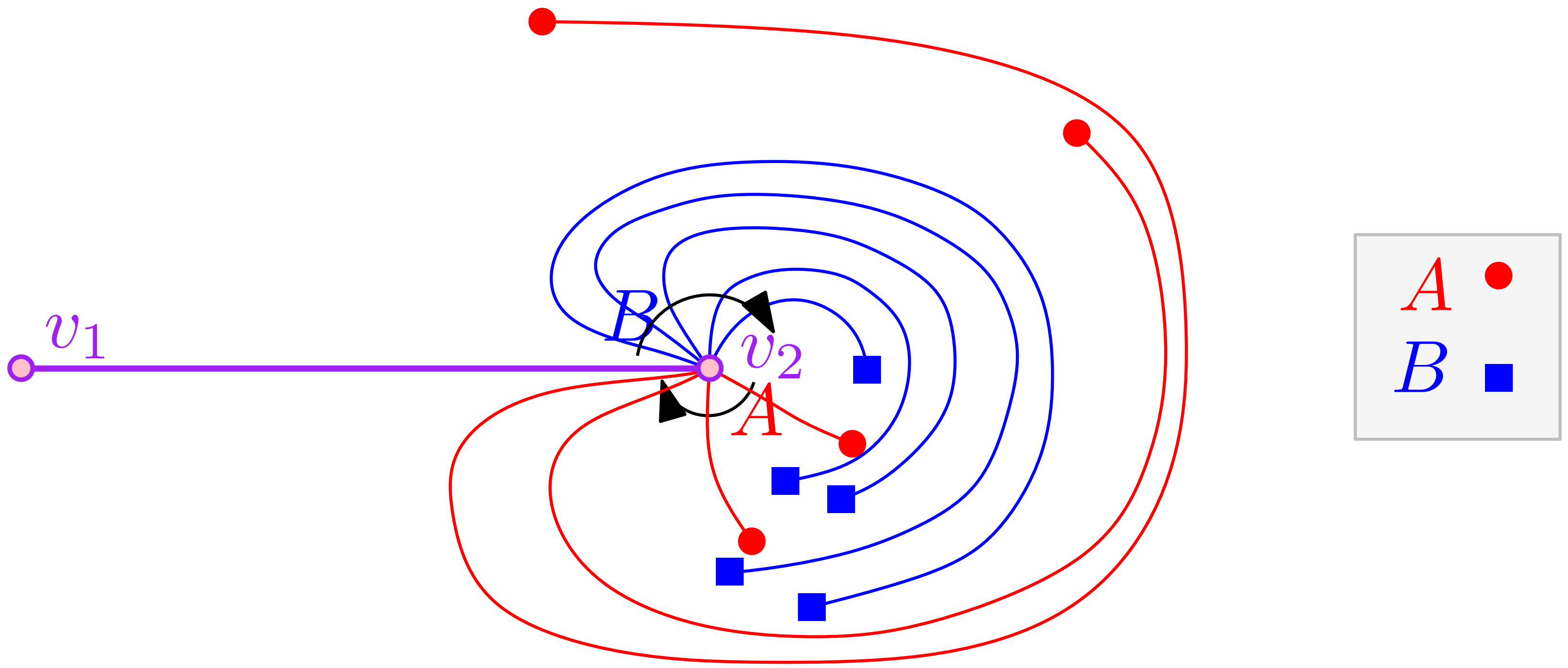

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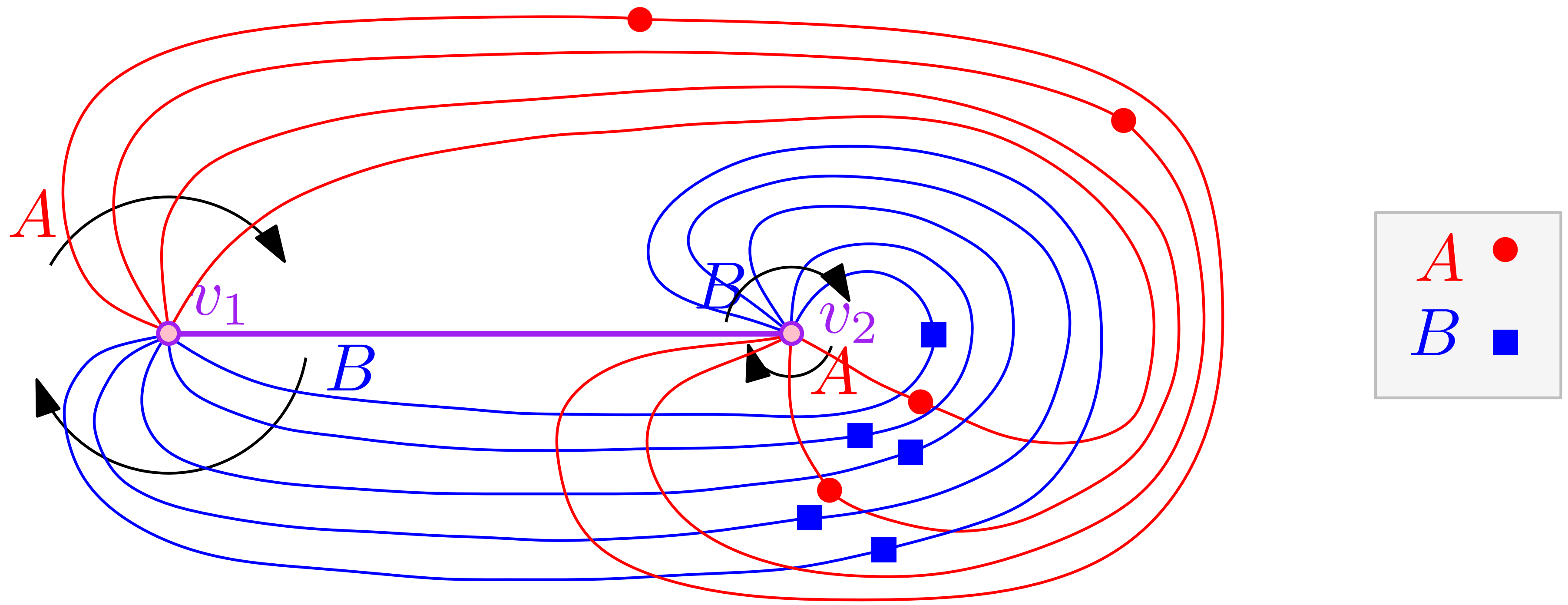

Recognizing Twistedness: Algorithmic Results

Realizable rotation system of K_n: $O(n^2)$-time algorithm to decide if generalized twisted

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Realizable rotation system of K_n: $O(n^2)$-time algorithm to decide if generalized twisted

Abstract rotation system of K_n: $O(n^5)$-time algorithm to decide if generalized twisted

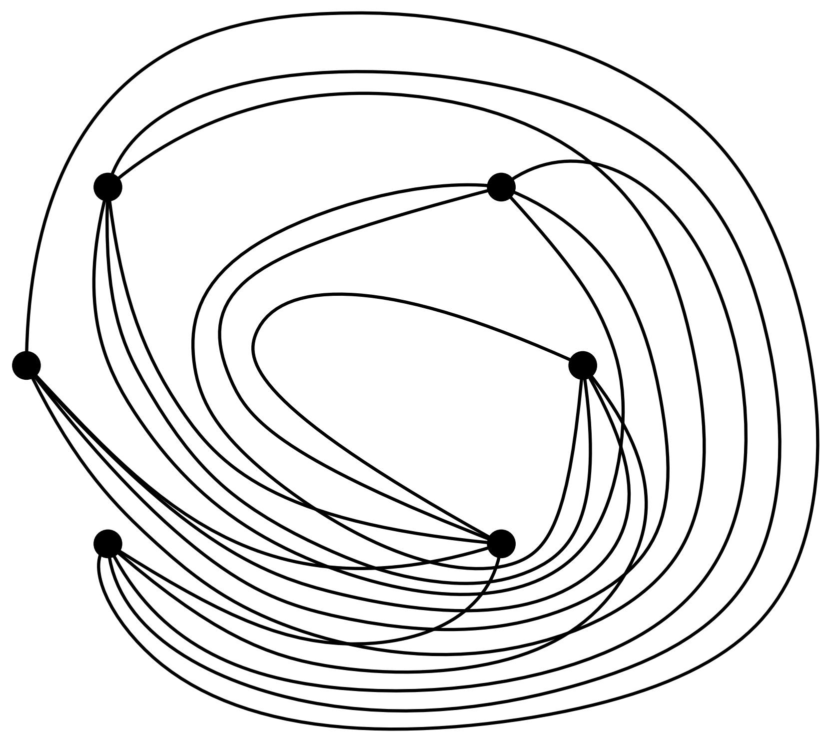
Algorithm

Realizable rotation system of K_n: $O(n^2)$-time algorithm to decide if generalized twisted

Abstract rotation system of K_n: $O(n^5)$-time algorithm to decide if generalized twisted

Algorithm

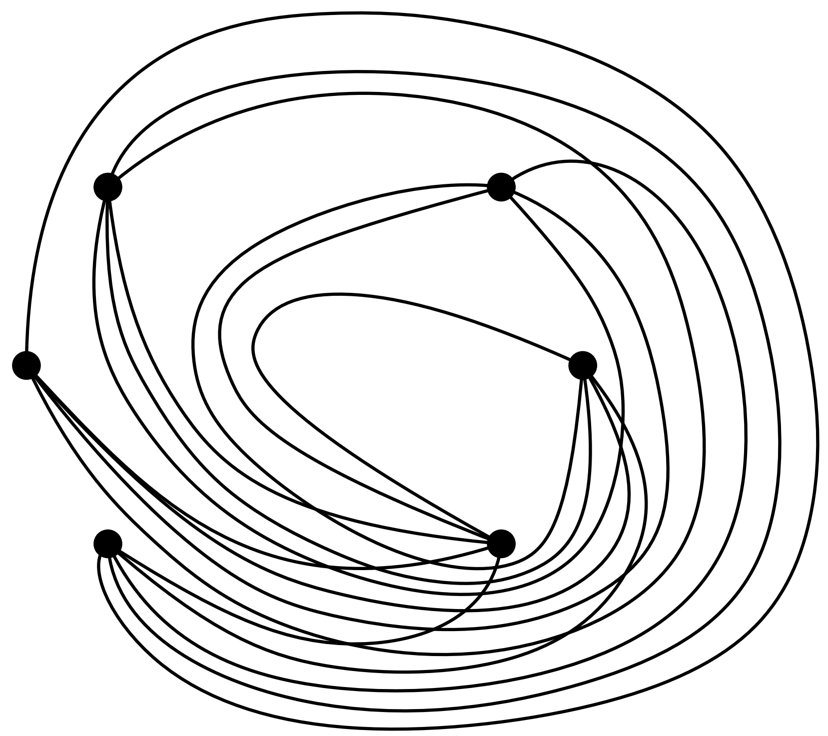
If rotation system realizable:



Algorithm

If rotation system realizable:

Step 1: For arbitrary vertex v compute all empty star triangles at vertex v. $O(n^2)$



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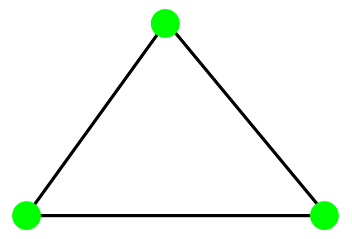
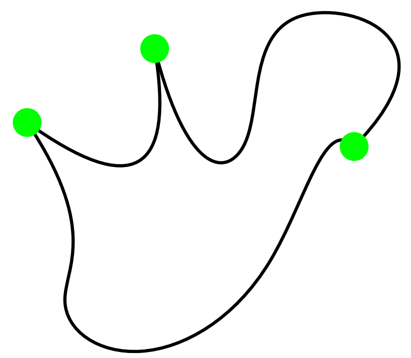
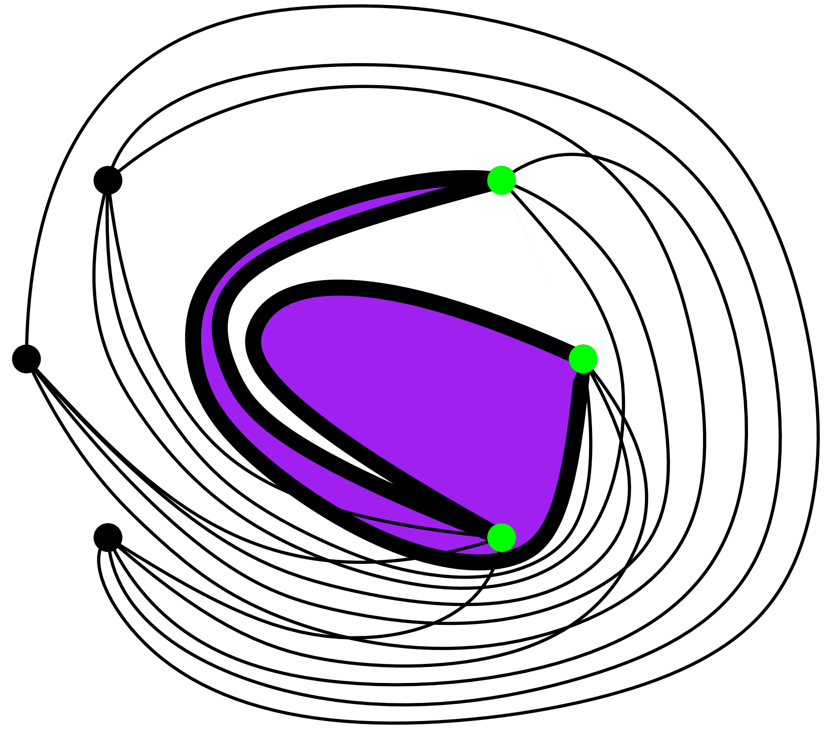
A triangle







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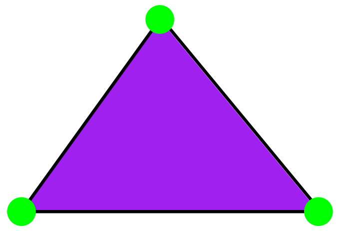

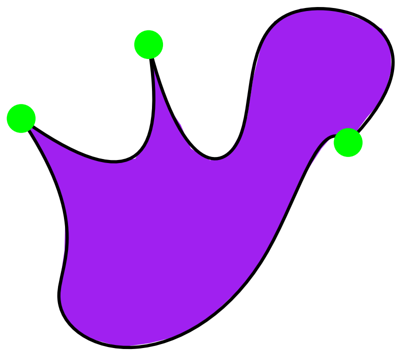

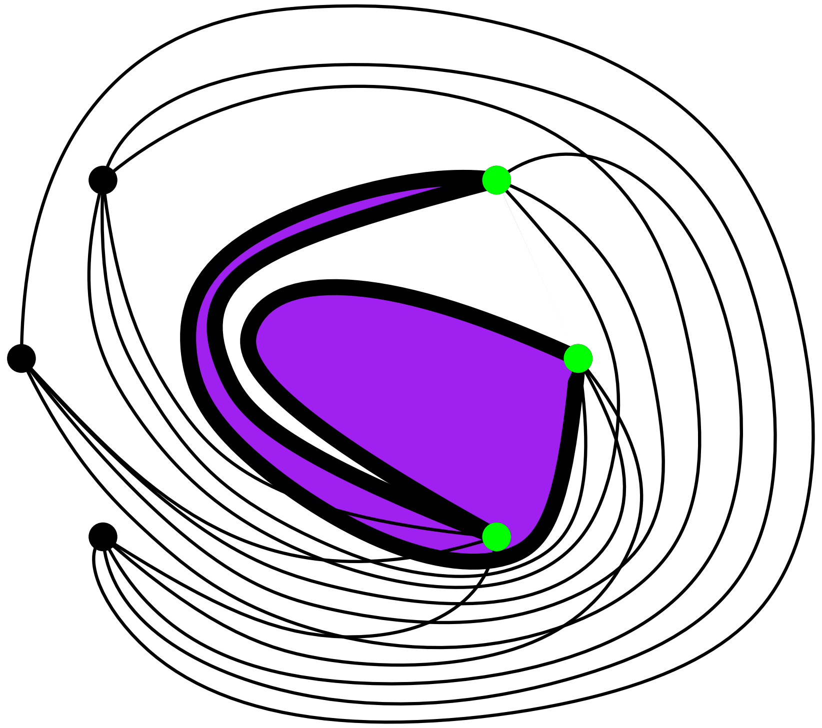

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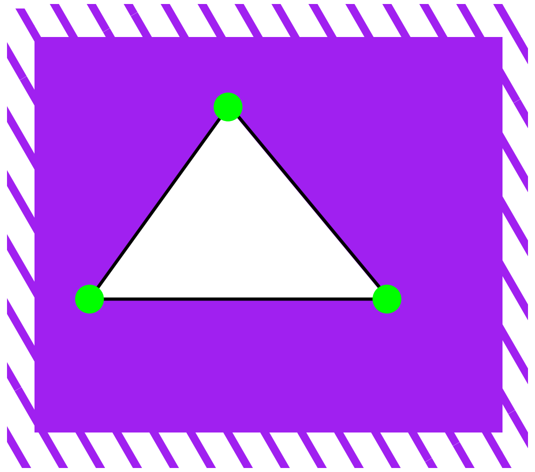

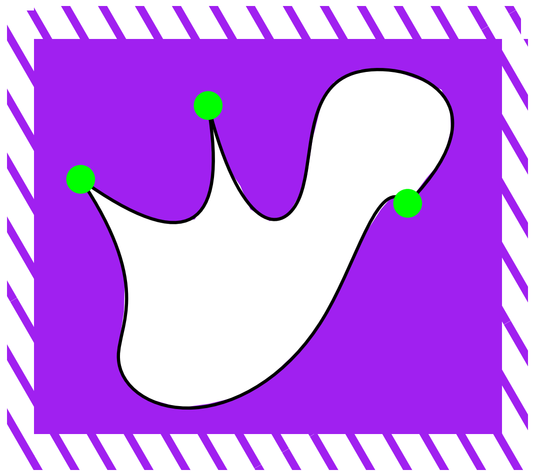

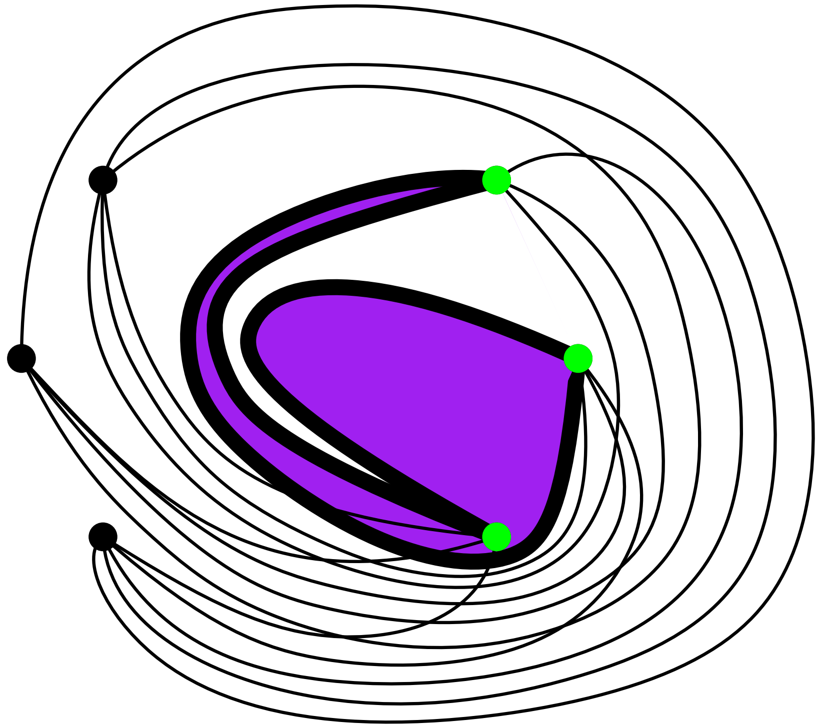

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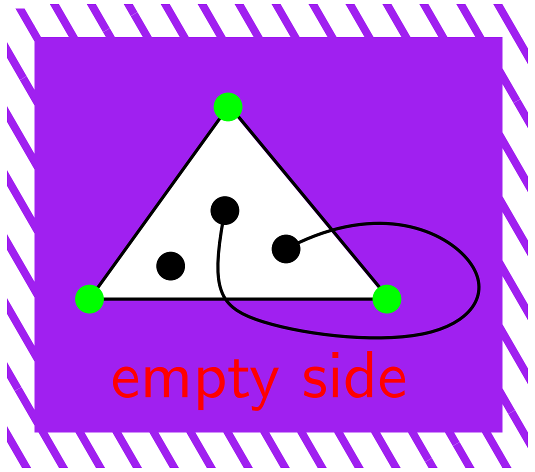
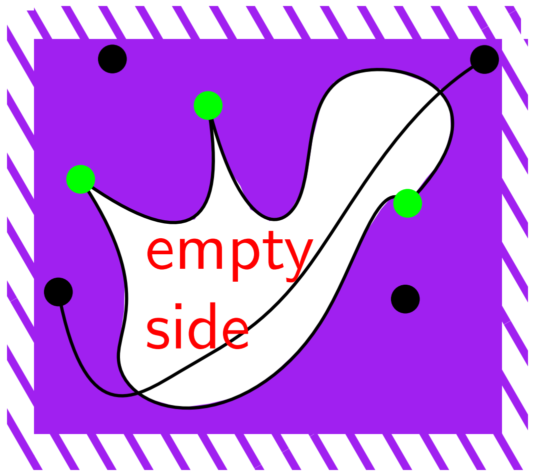
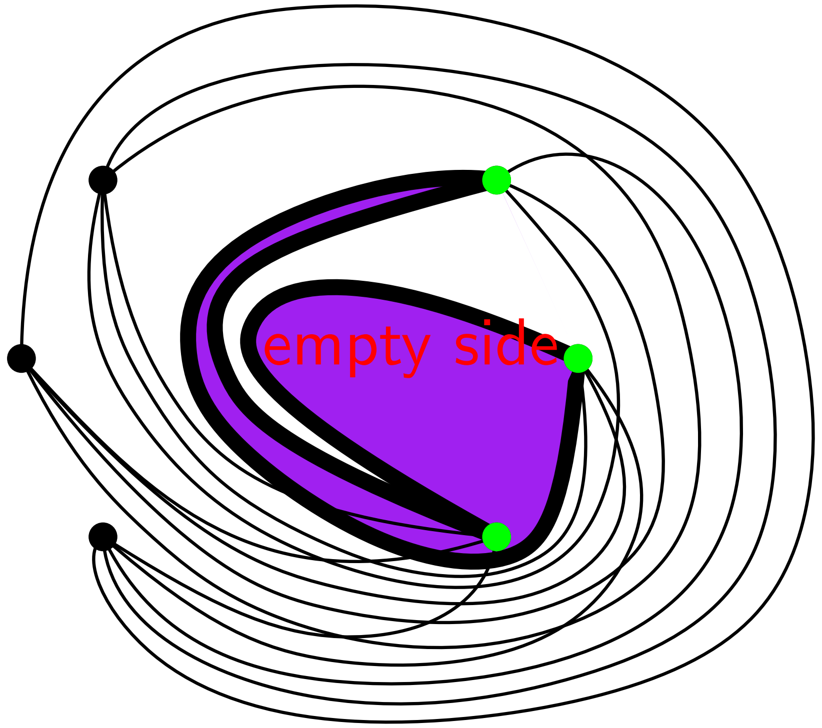
A triangle is empty if at least one side of the triangle contains no vertices.







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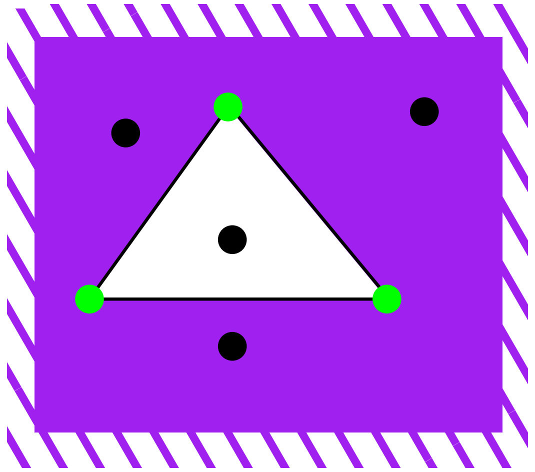

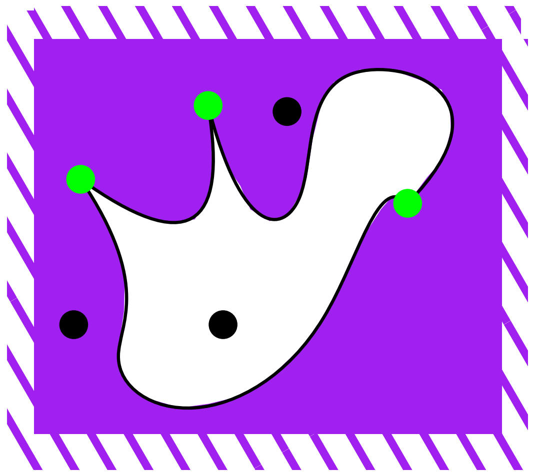

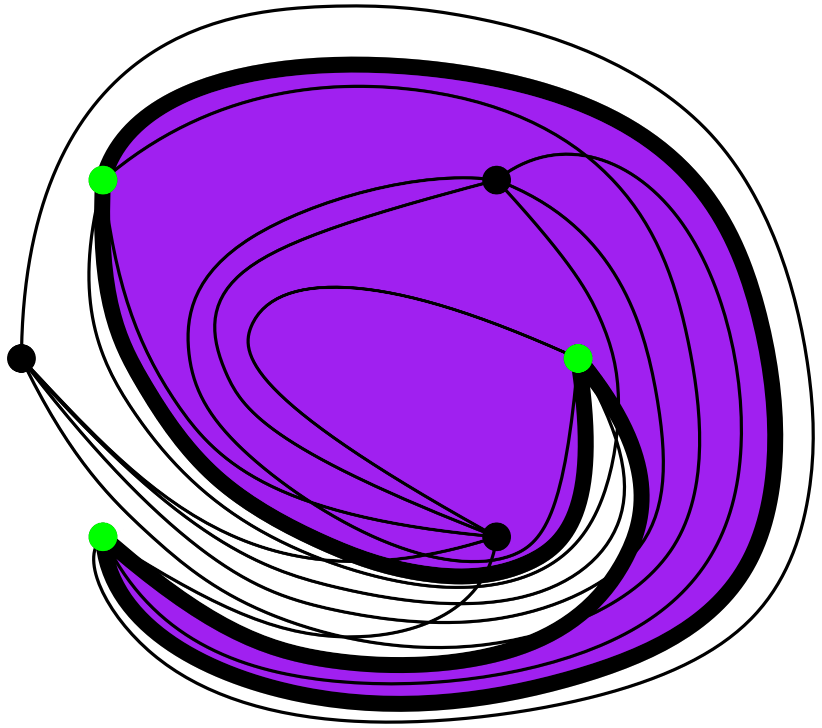

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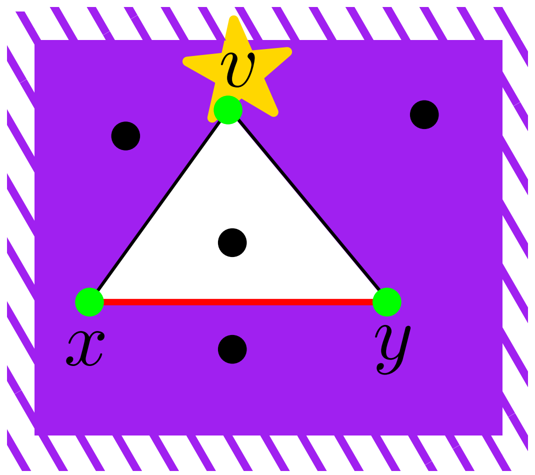
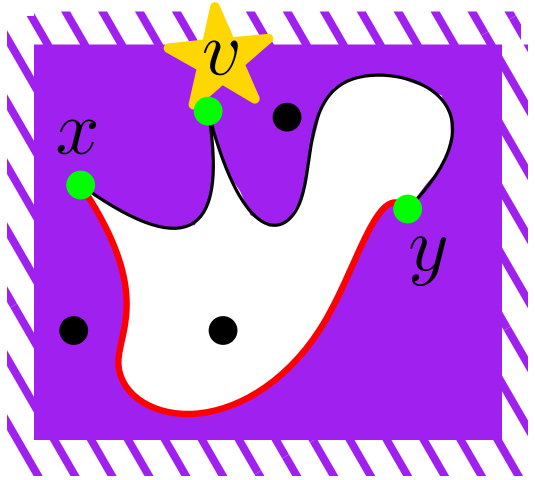
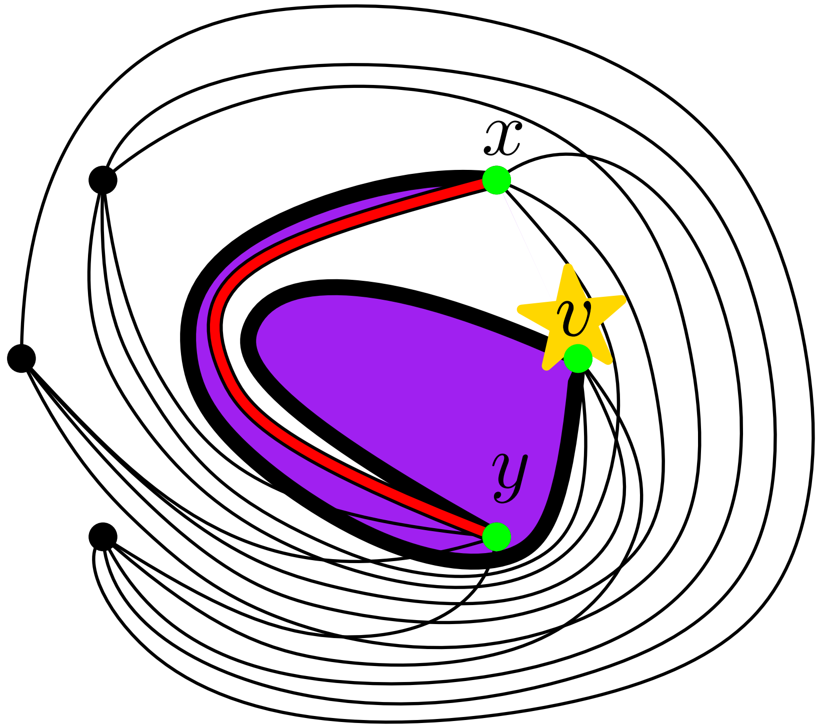
A triangle with vertices v, x, y is a star triangle at v if xy is not crossed by any edges incident to v.







Algorithm

If rotation system realizable:

Step 1: For arbitrary vertex v compute all empty star triangles at vertex v. $O(n^2)$

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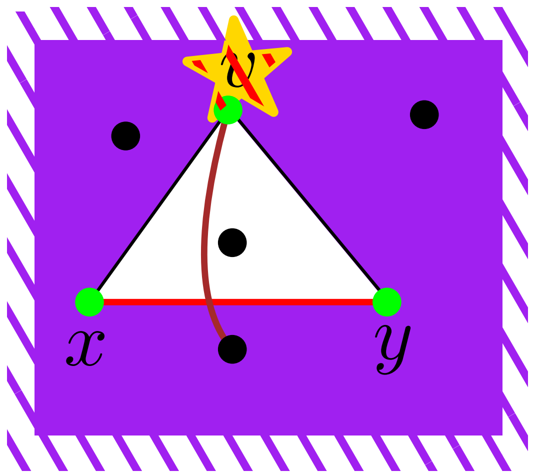

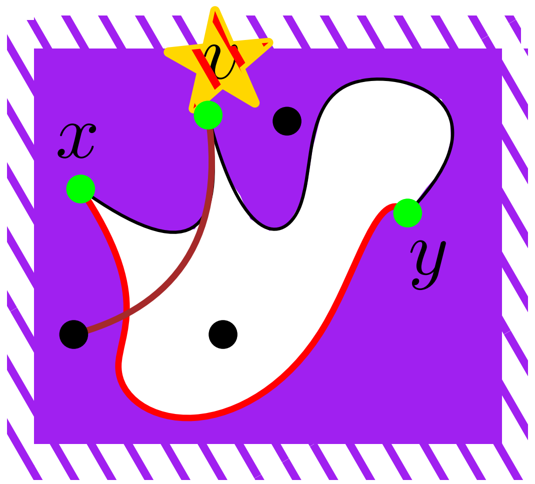

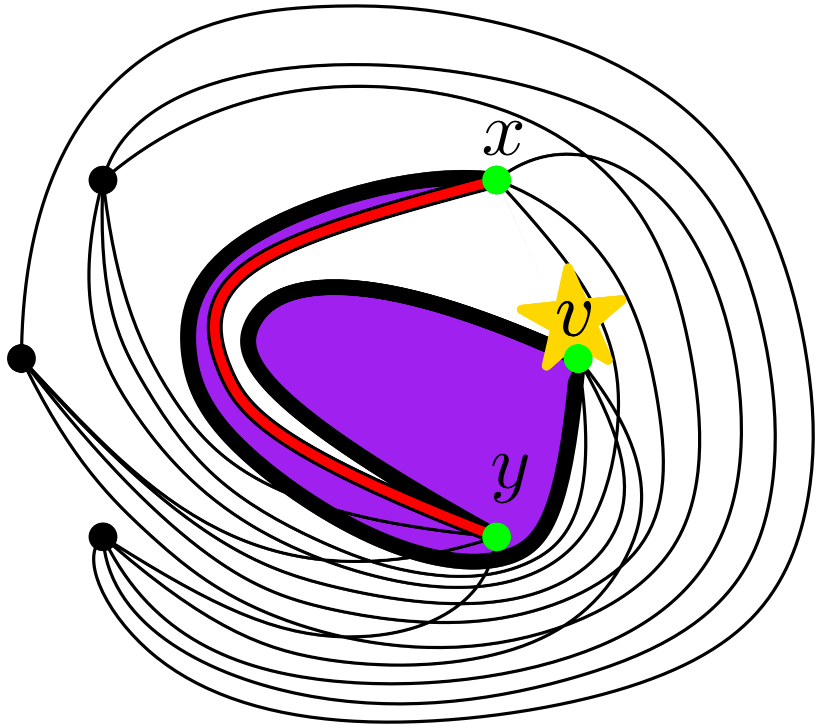

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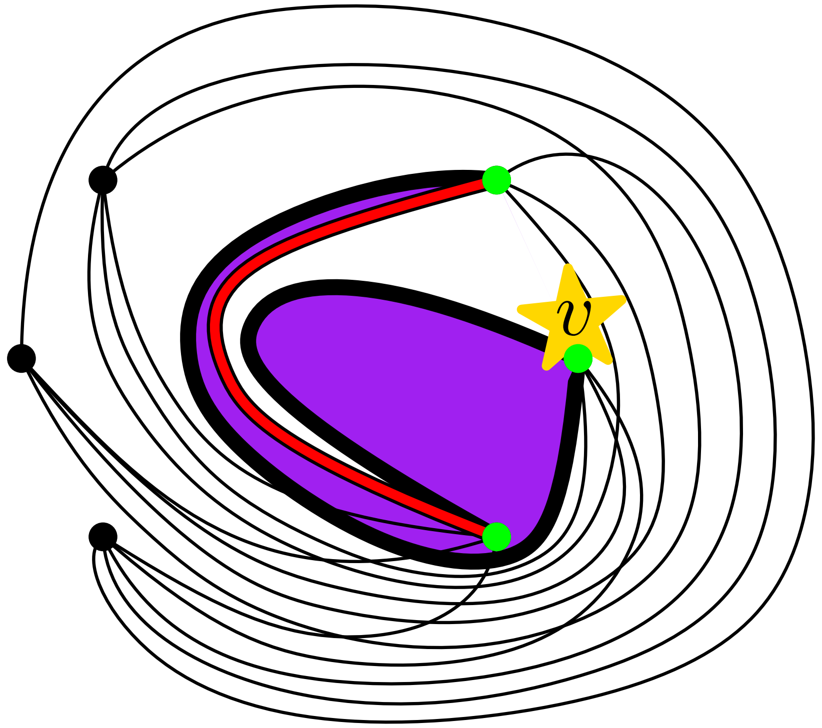


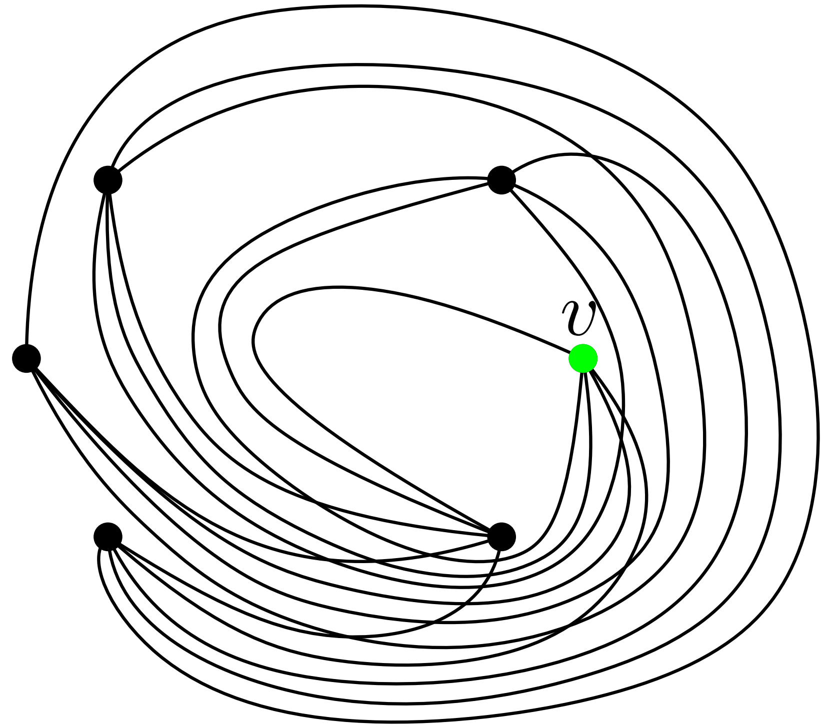
Algorithm

If rotation system realizable:

Step 1: For arbitrary vertex v compute all empty star triangles at vertex v. $O(n^2)$

Step 2:

- If there were **not** exactly 2 empty star triangles at v: not generalized twisted [García, Tejel, Vogtenhuber, W. 2022]



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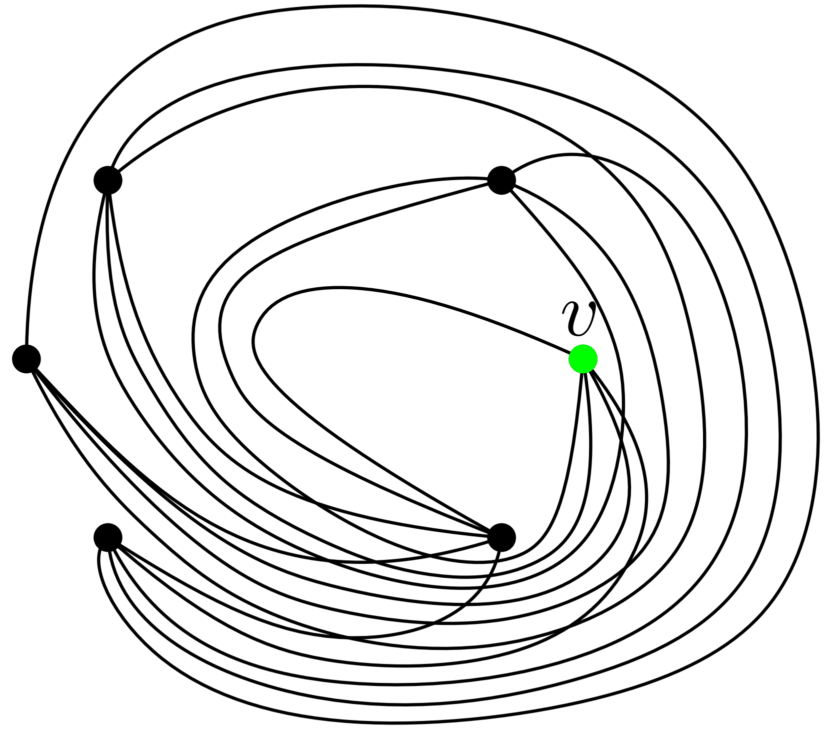

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two vertices v_1 and v_2 and a bipartition $A \cup B$ of the remaining vertices (A or B can be empty) such that:

- $\forall$ a, a' in A, the edge aa' crosses $v_1 v_2$.
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- $\forall$ $a \in A$, $b \in B$, the edge ab does not cross $v_1 v_2$.
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Algorithm

Realizable rotation system of K_n: $O(n^2)$-time algorithm to decide if generalized twisted

Abstract rotation system of K_n: $O(n^5)$-time algorithm to decide if generalized twisted

If rotation system realizable:

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Conclusion

An abstract rotation system of K_n with $n \geq 7$ is generalized twisted
$\Leftrightarrow$ every subrotation system induced by 5 vertices is.

A realizable rotation system of K_n with $n \geq 3$ is generalized twisted $\Leftrightarrow$
there exist two vertices v_1 and v_2 and a bipartition $A \cup B$ of the remaining vertices (A or B can be empty) s.t.:

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Realizable rotation system of K_n: $O(n^2)$-time algorithm to decide if generalized twisted
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Slides created with IPE:
Thanks to Otfried Cheong!

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Realizable rotation system of K_n: $O(n^2)$-time algorithm to decide if generalized twisted
Abstract rotation system of K_n: $O(n^5)$-time algorithm to decide if generalized twisted

Next Step: Use generalized twisted drawings
One possible direction: Does every simple drawing of K_n contain
a large generalized twisted drawing or large (generalized) convex drawing?

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Conclusion

Thank you for listening!

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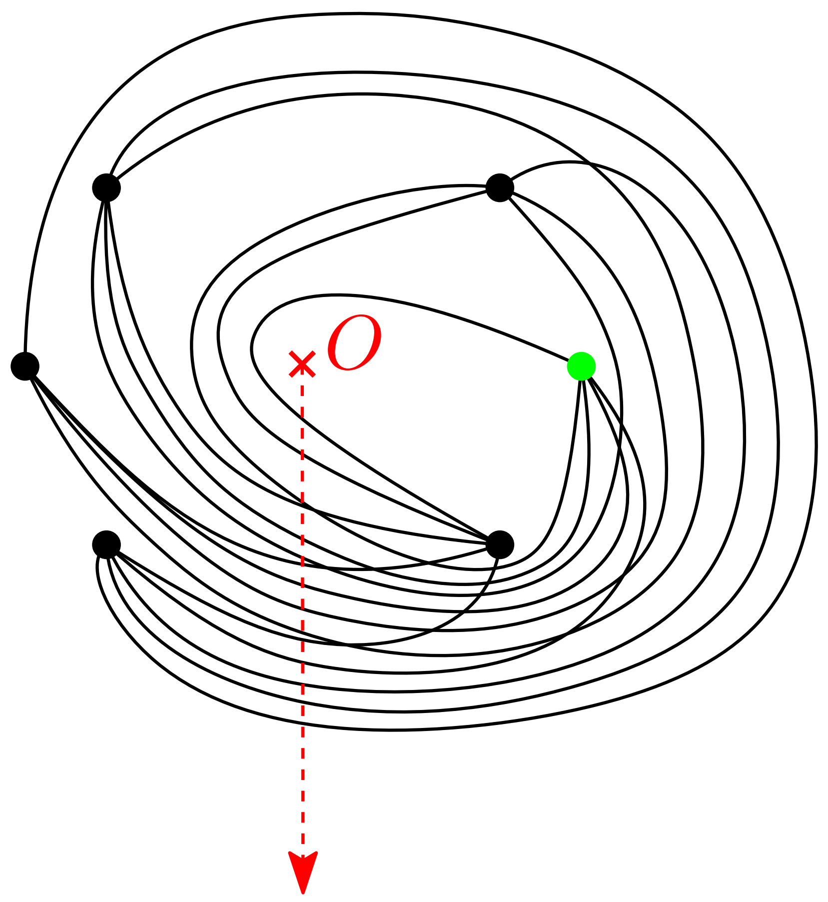
Crucial Proof Element: Antipodal Vi-Cells

Simple drawing D of K_n is generalized twisted

$\Leftrightarrow$

$\exists$ a pair of antipodal vi-cells [Aichholzer, García, Tejel, Vogtenhuber, W. 2022/2024]



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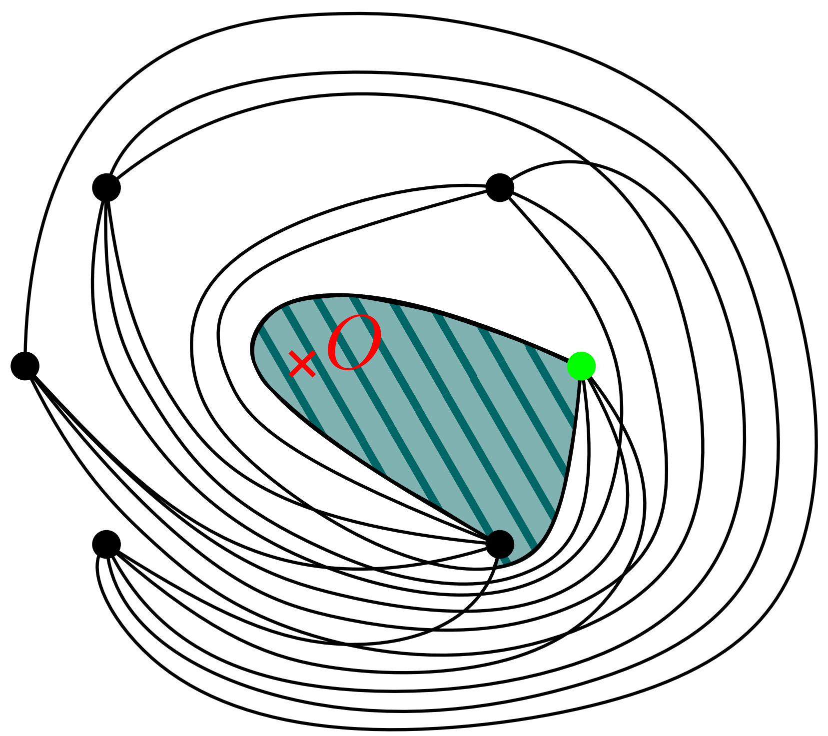
vi-cell ... cell with a vertex on its boundary



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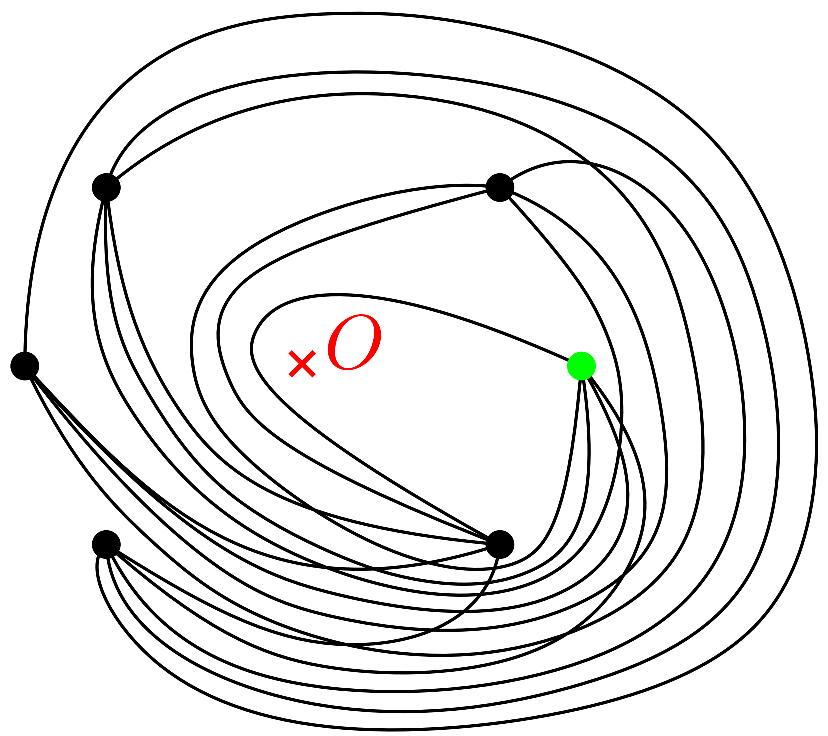

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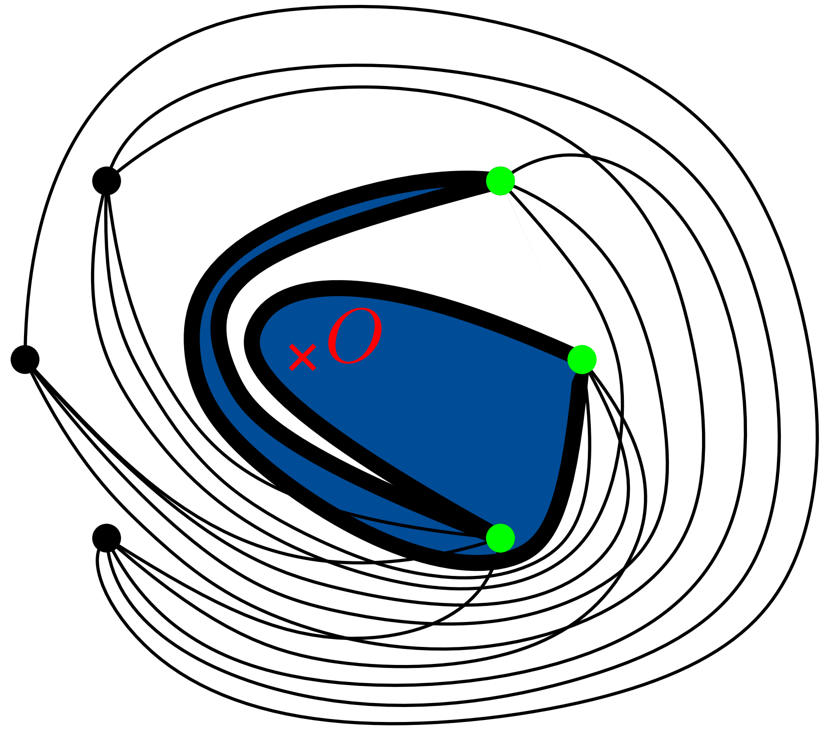

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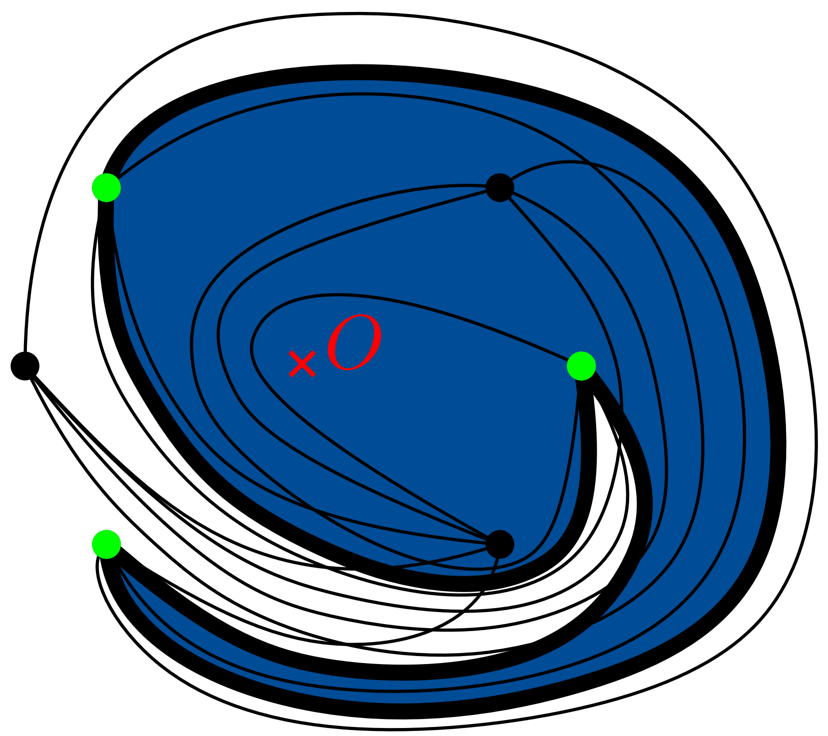

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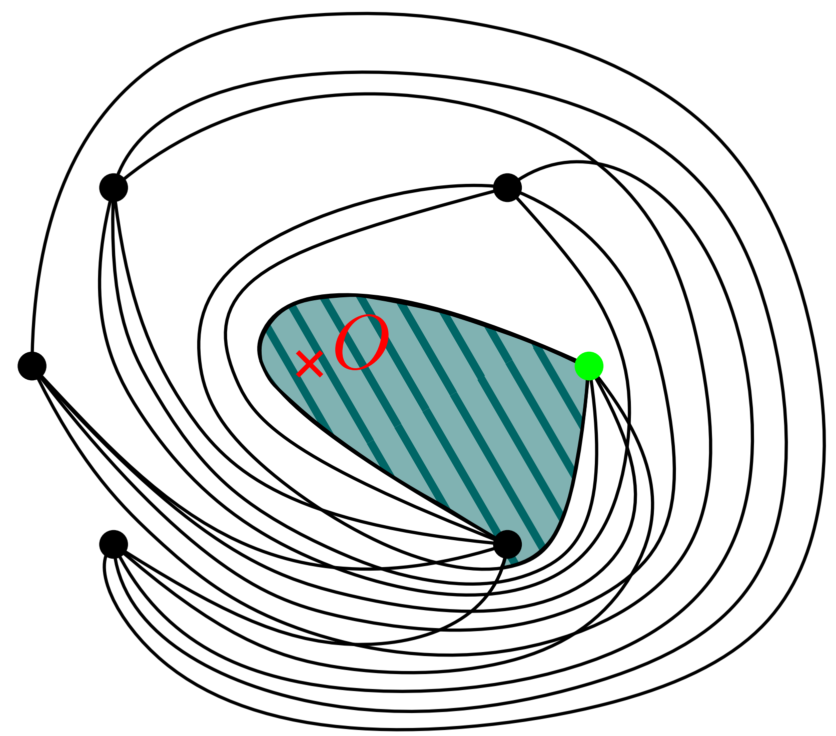

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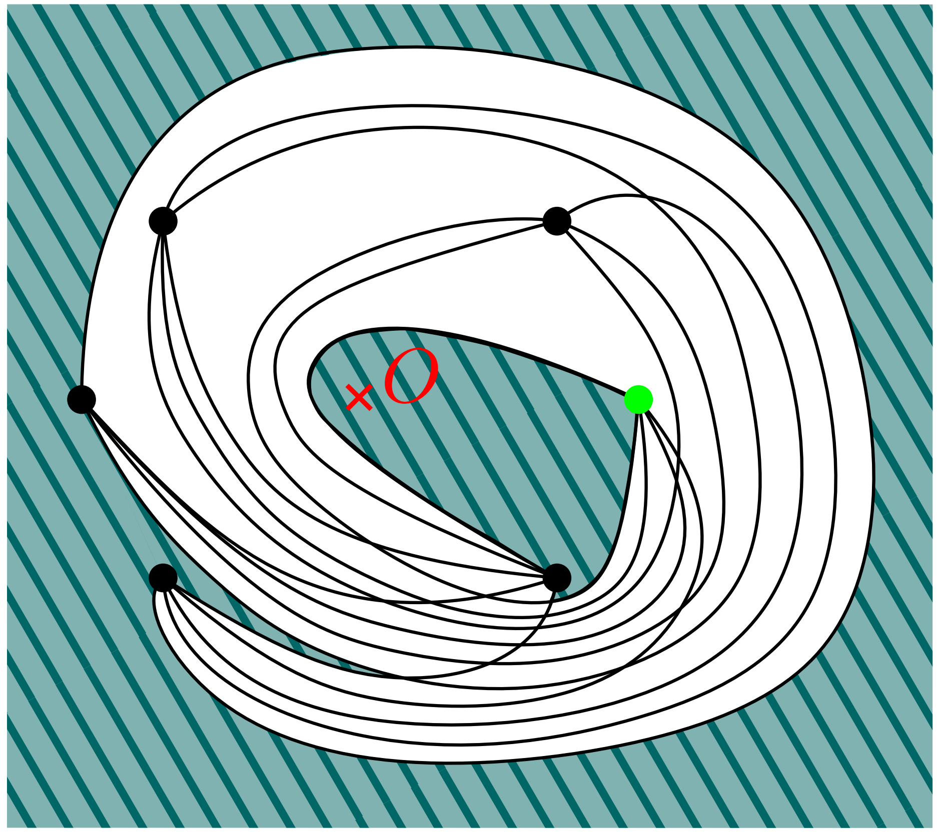

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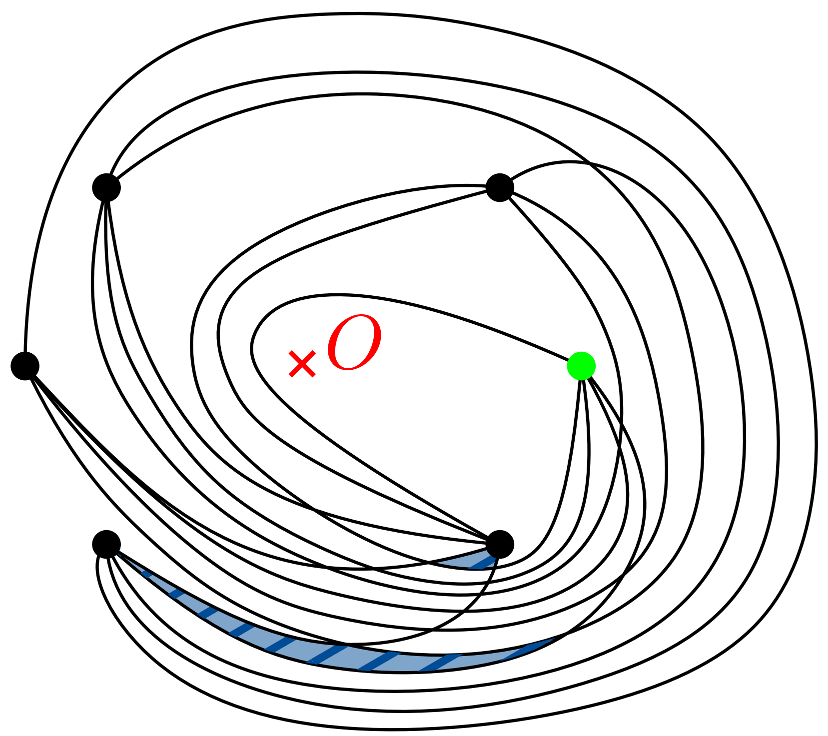

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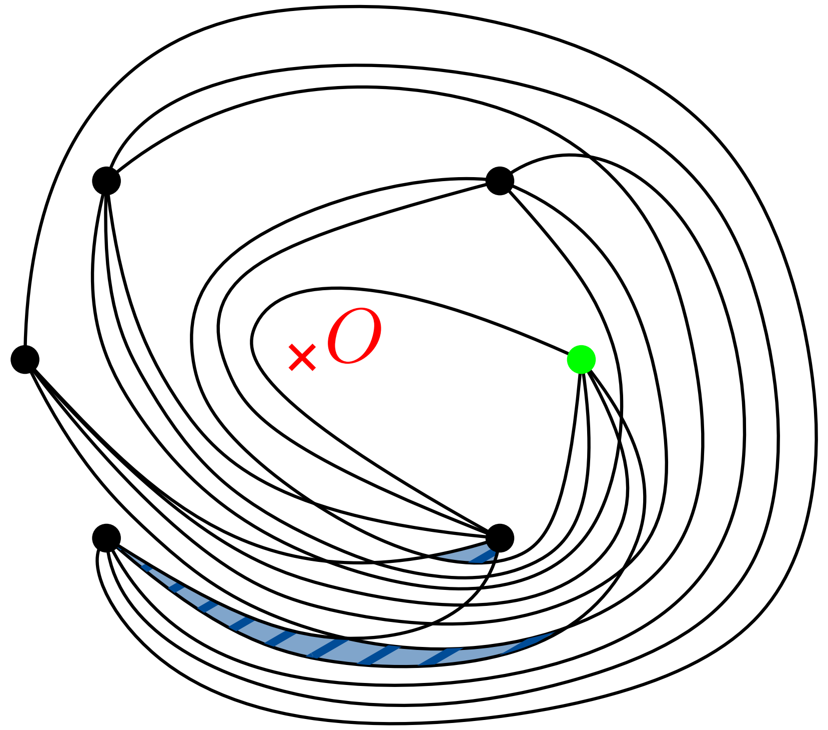

At most two pairs of antipodal vi-cells

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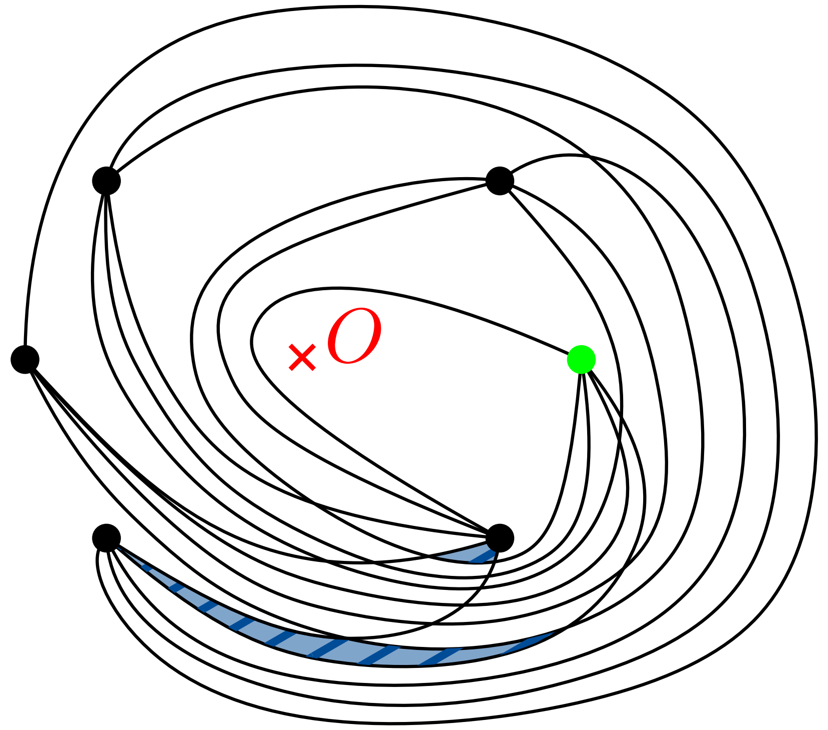

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If there are two, they are adjacent along an edge (crossing all it can)

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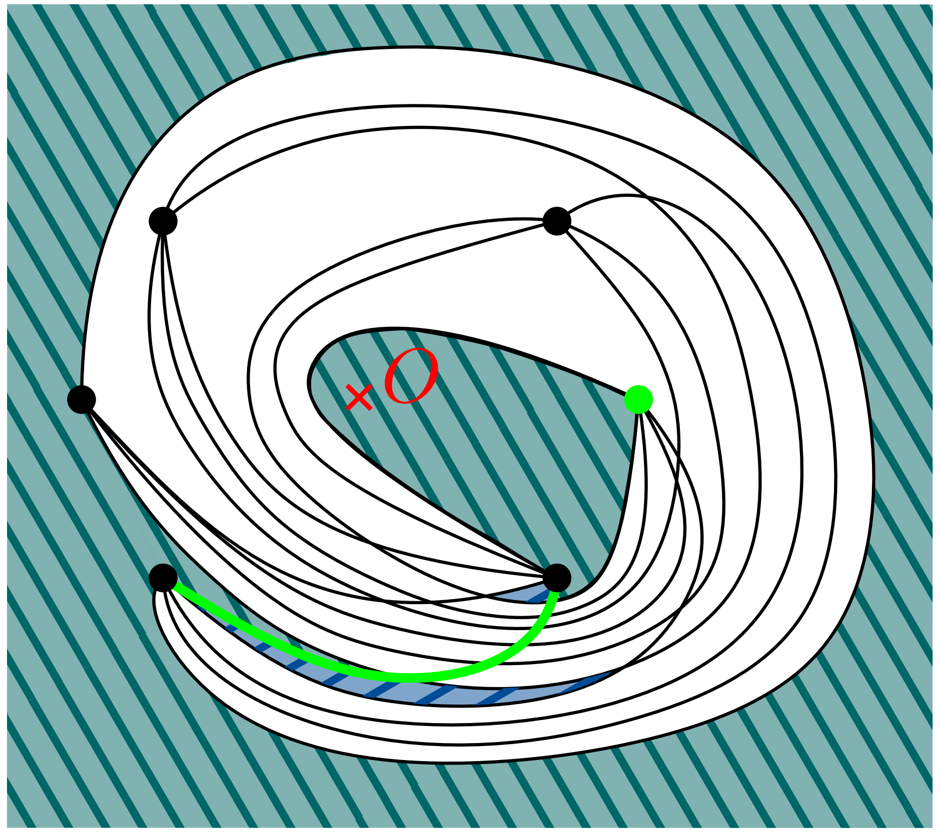

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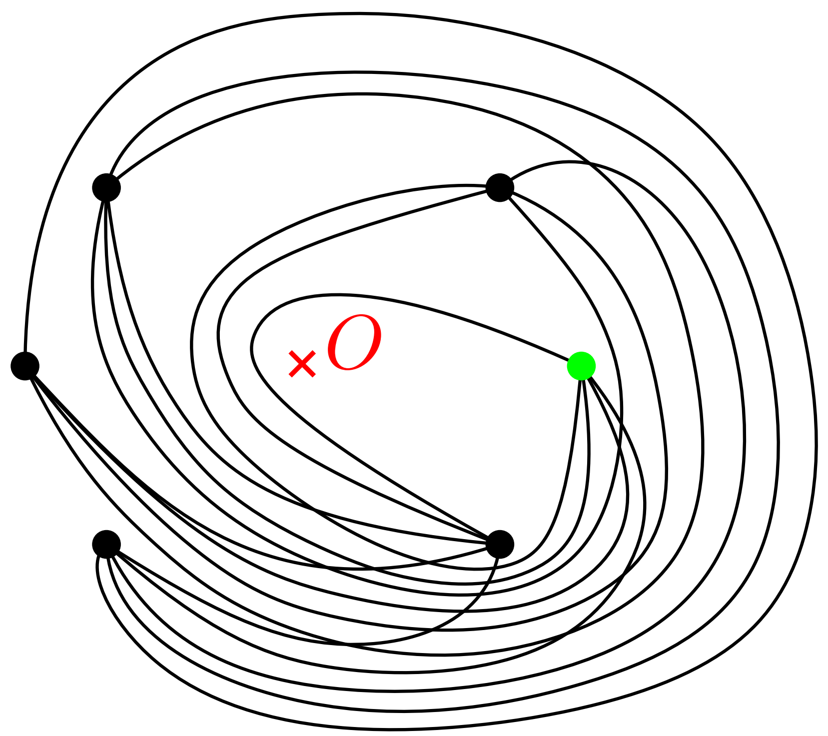

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Antipodal vi-cells of drawing are inside antipodal vi-cells of subdrawing

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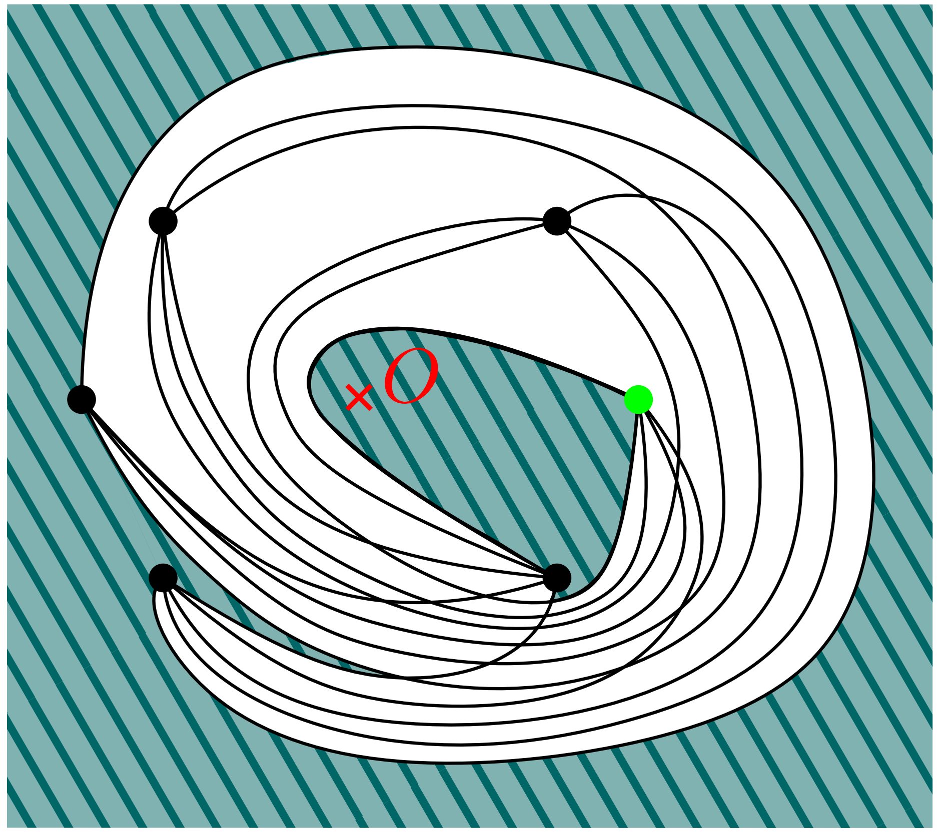

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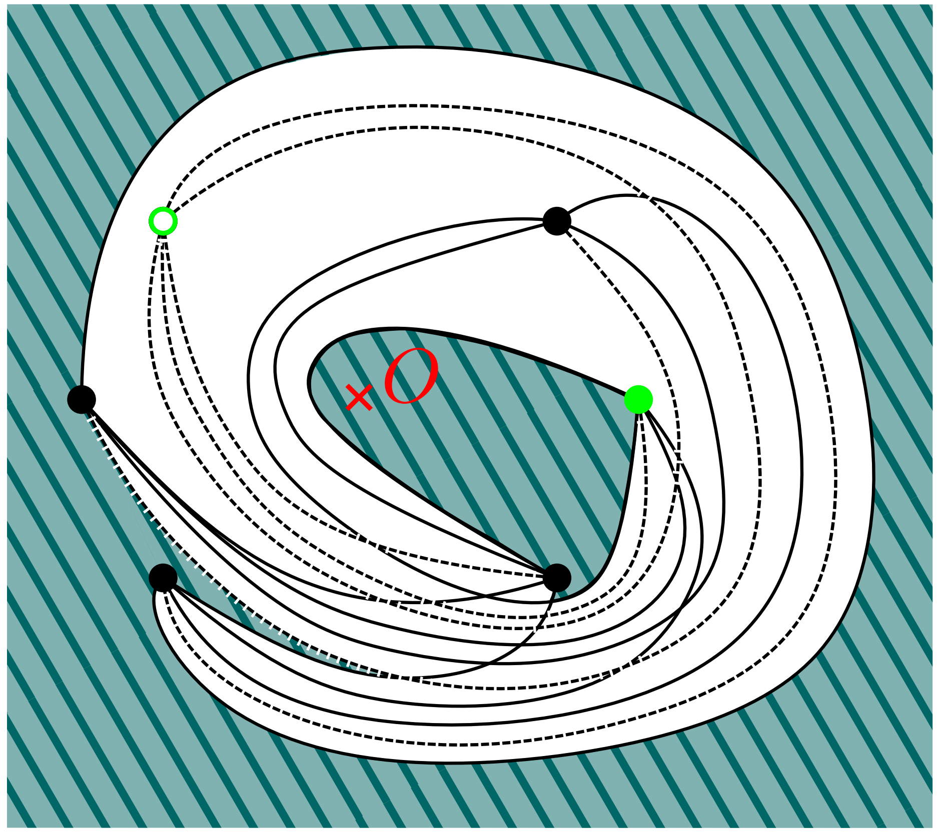

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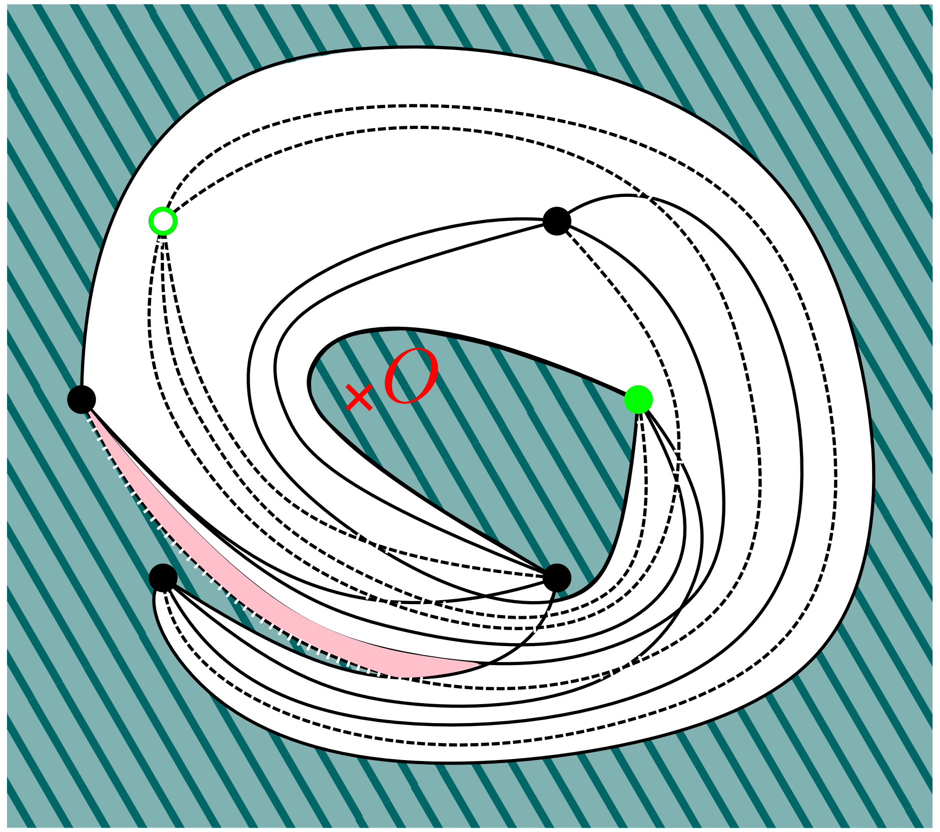

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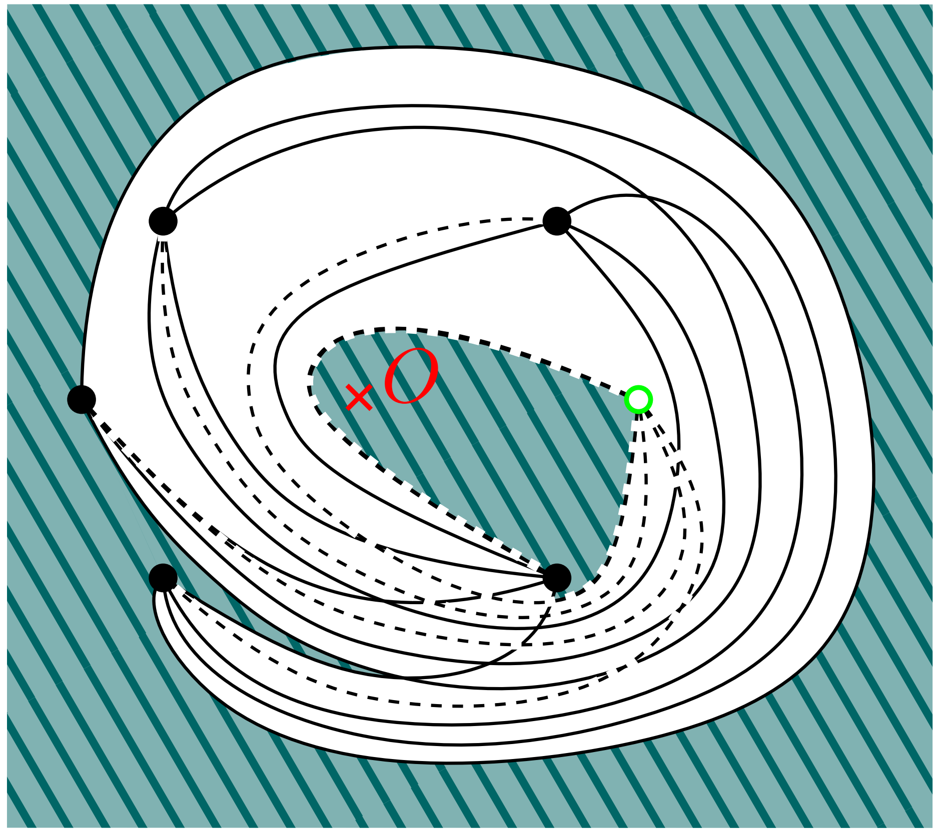

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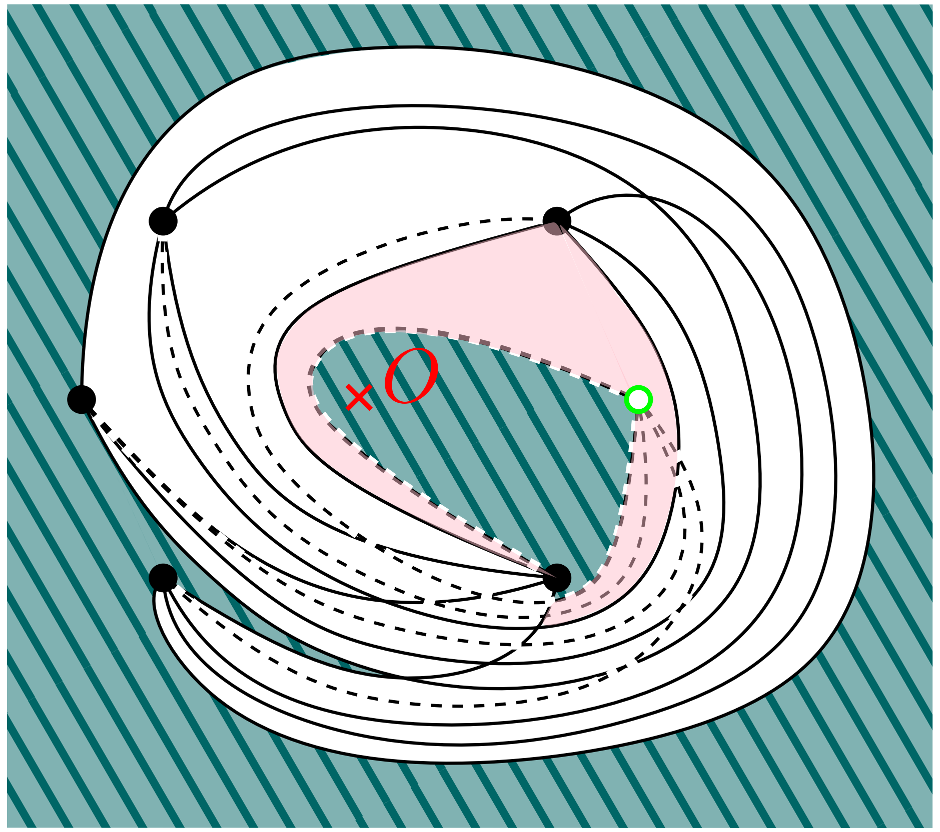

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Characterization via 5-Tuple: Glimpse at Proof Sketch

An abstract rotation system of K_n with $n \geq 7$ is generalized twisted
$\Leftrightarrow$ every subrotation system induced by 5 vertices is.

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Proof by induction

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Verification with computer for $7 \leq n \leq 11$

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Proof by induction

Verification with computer for $7 \leq n \leq 11$

By hypothesis every rotations system induced by $\leq n-1$ vertices is generalized twisted.

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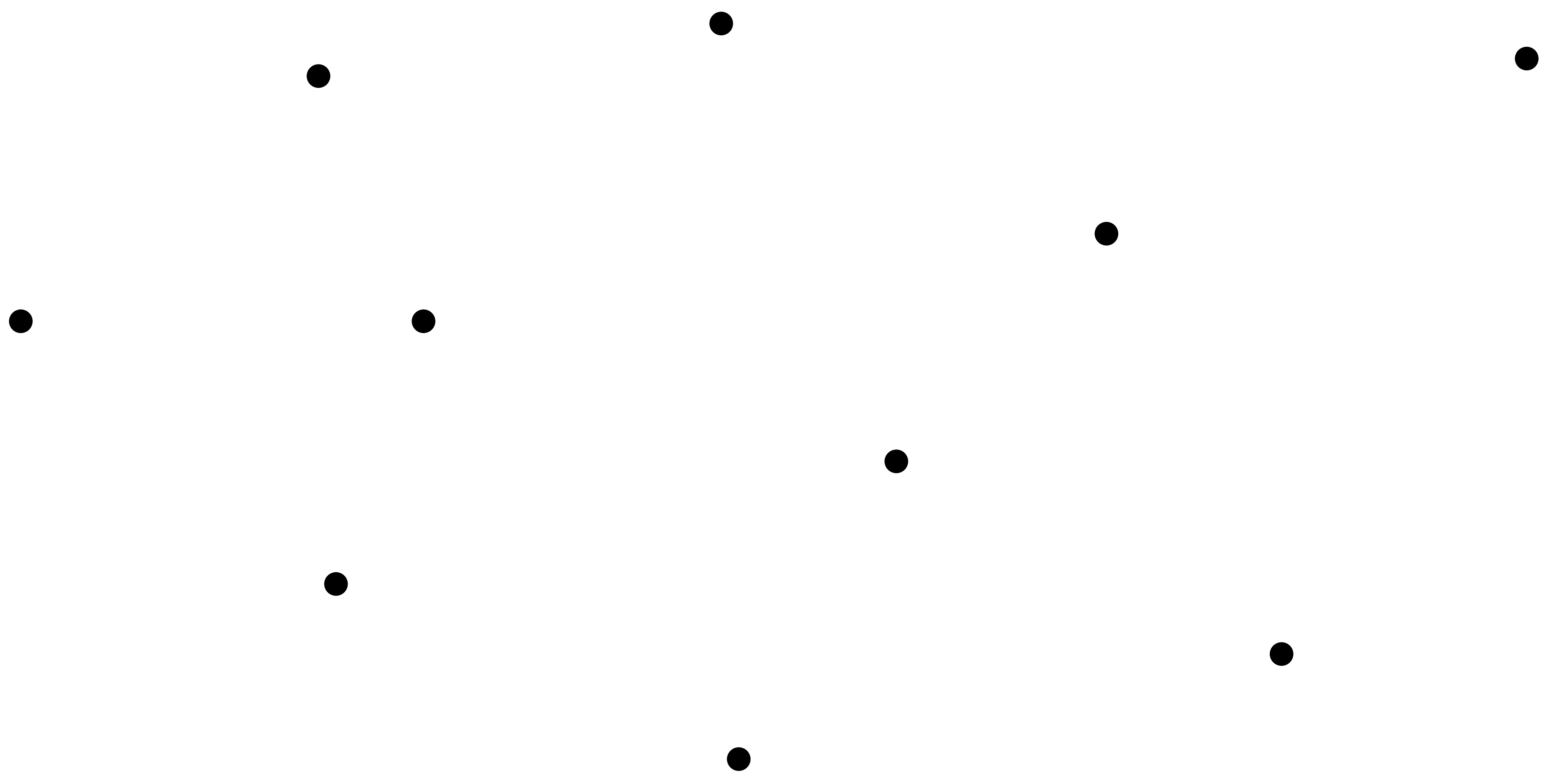

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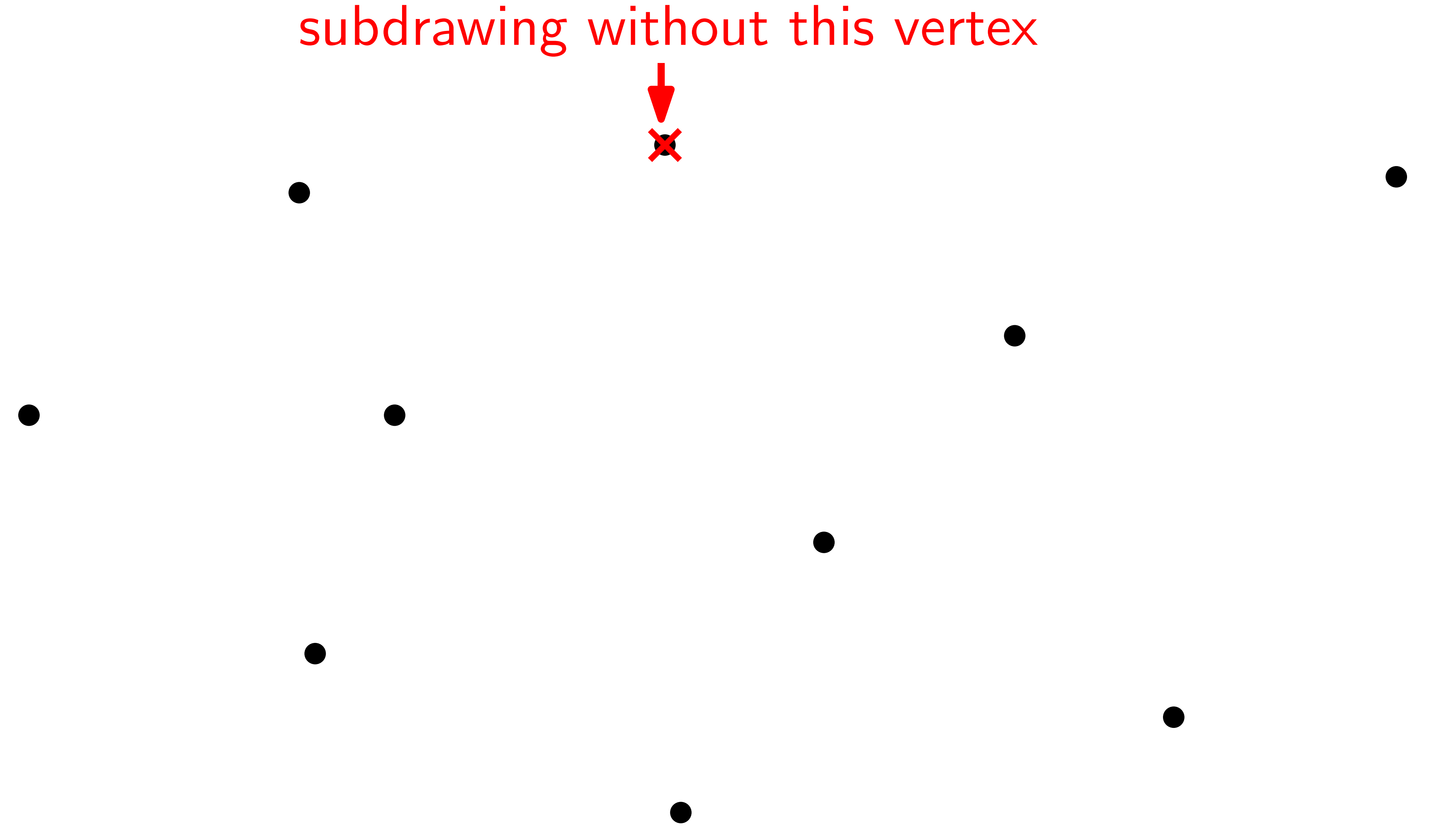

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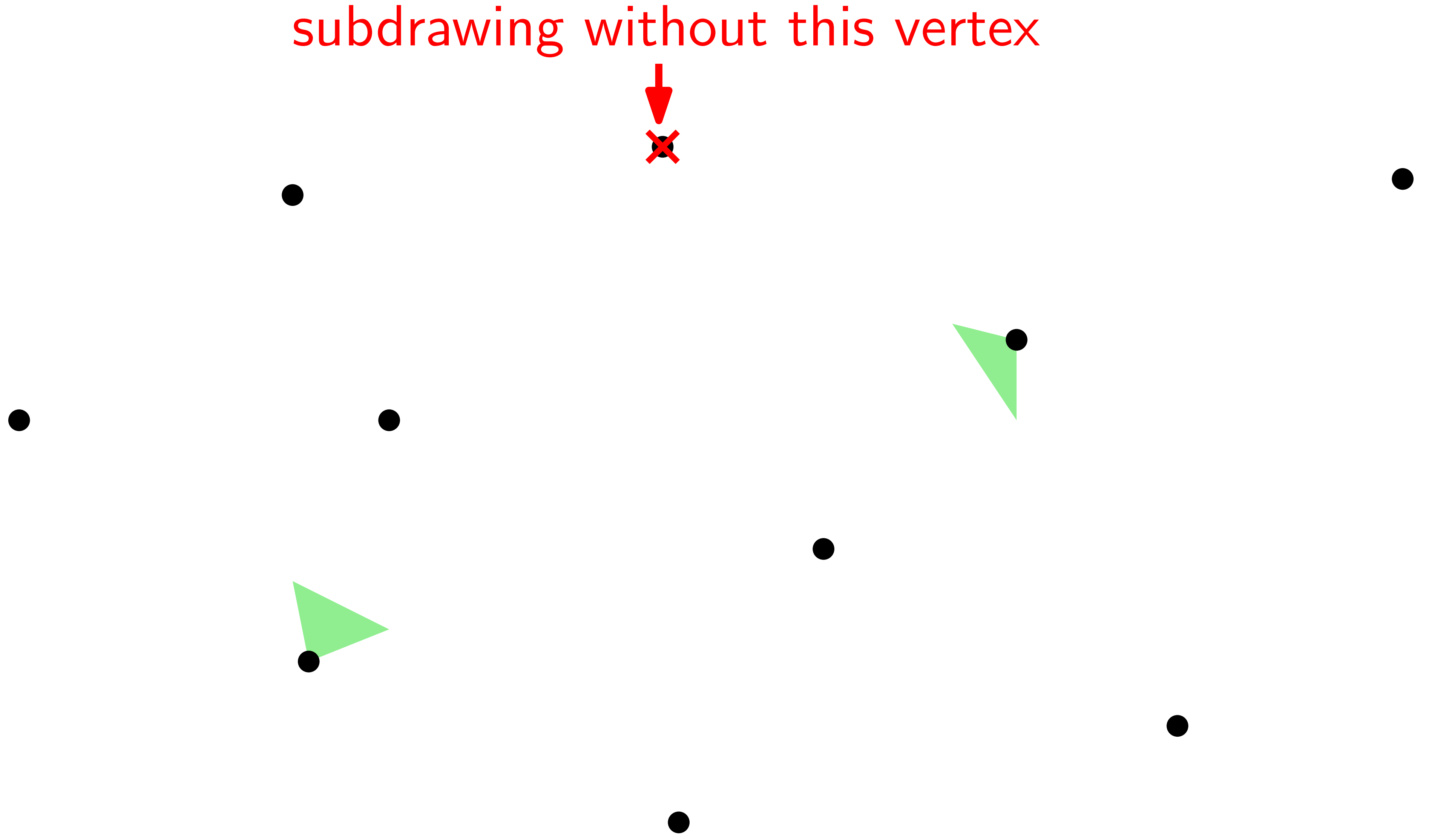

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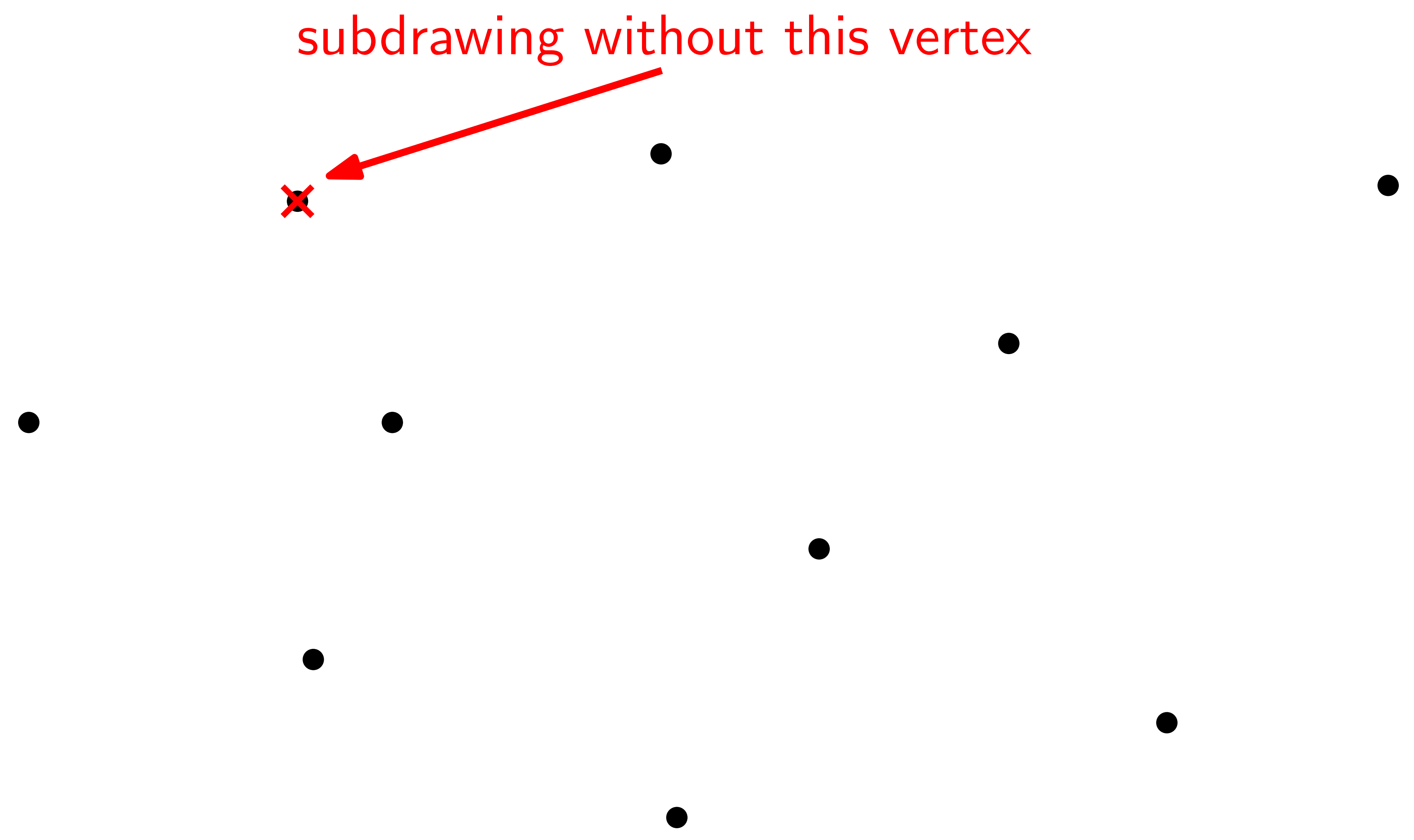

Characterization via 5-Tuple: Glimpse at Proof Sketch

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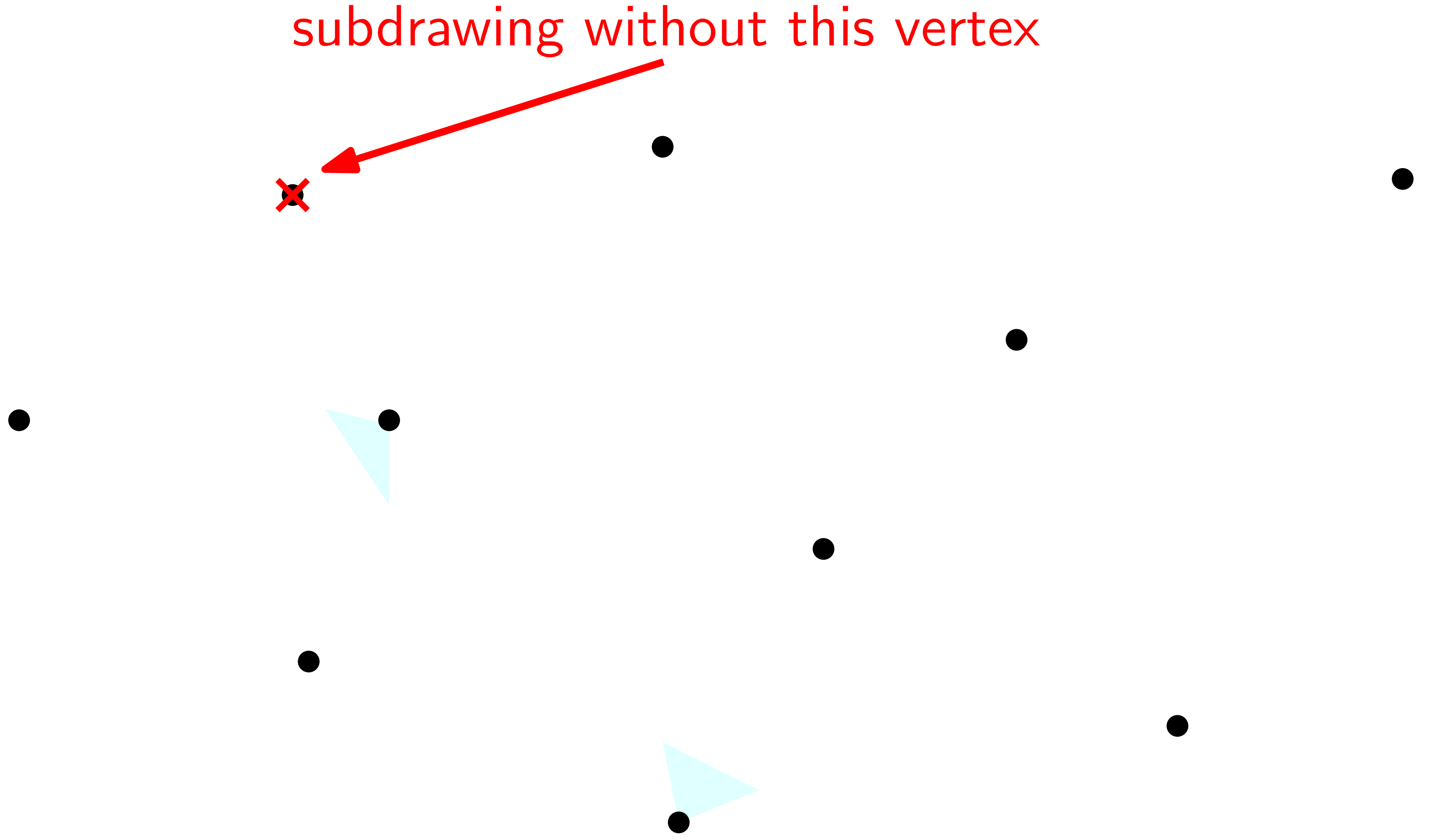
$\Leftrightarrow$ every subrotation system induced by 5 vertices is.



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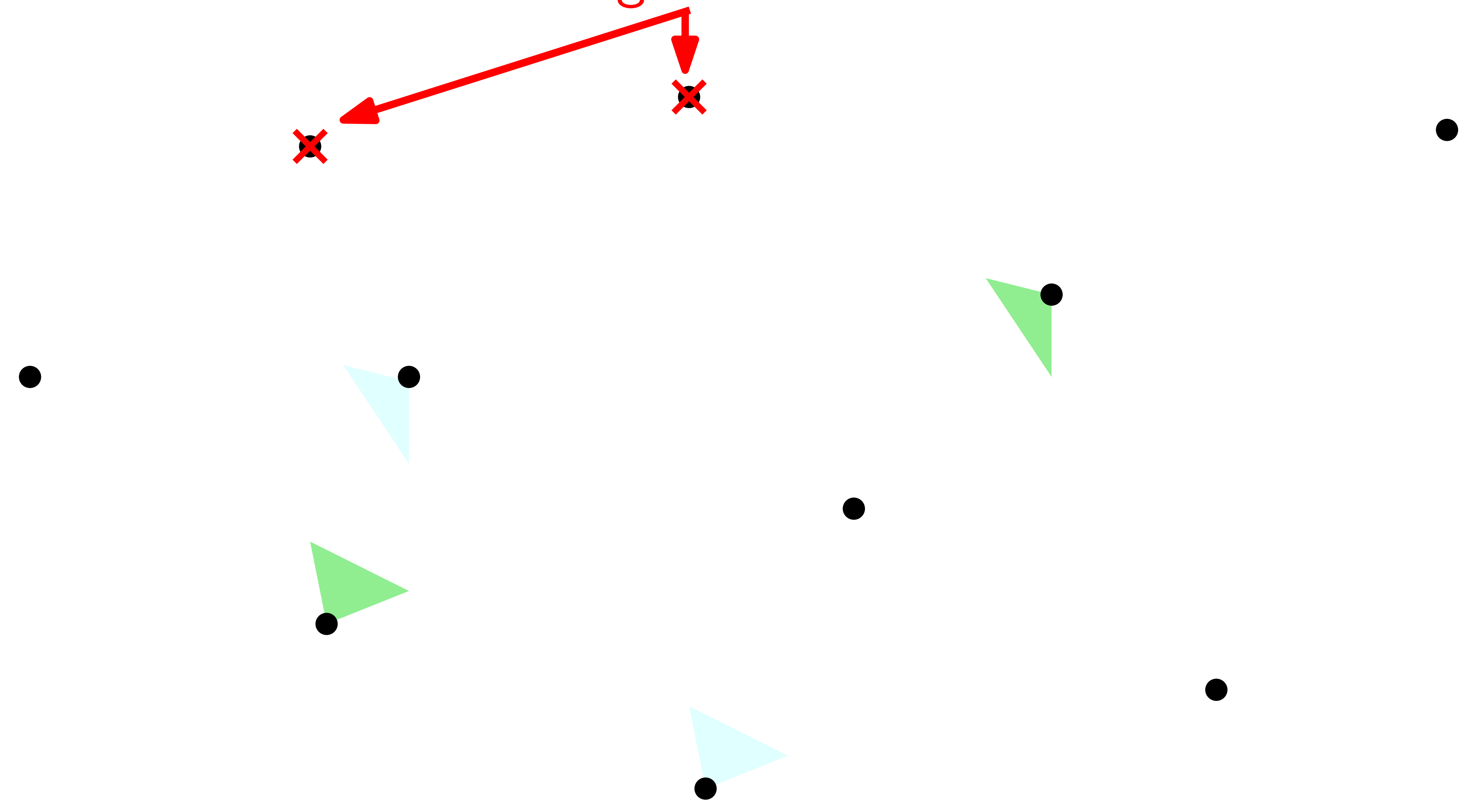
subdrawing without both these vertices



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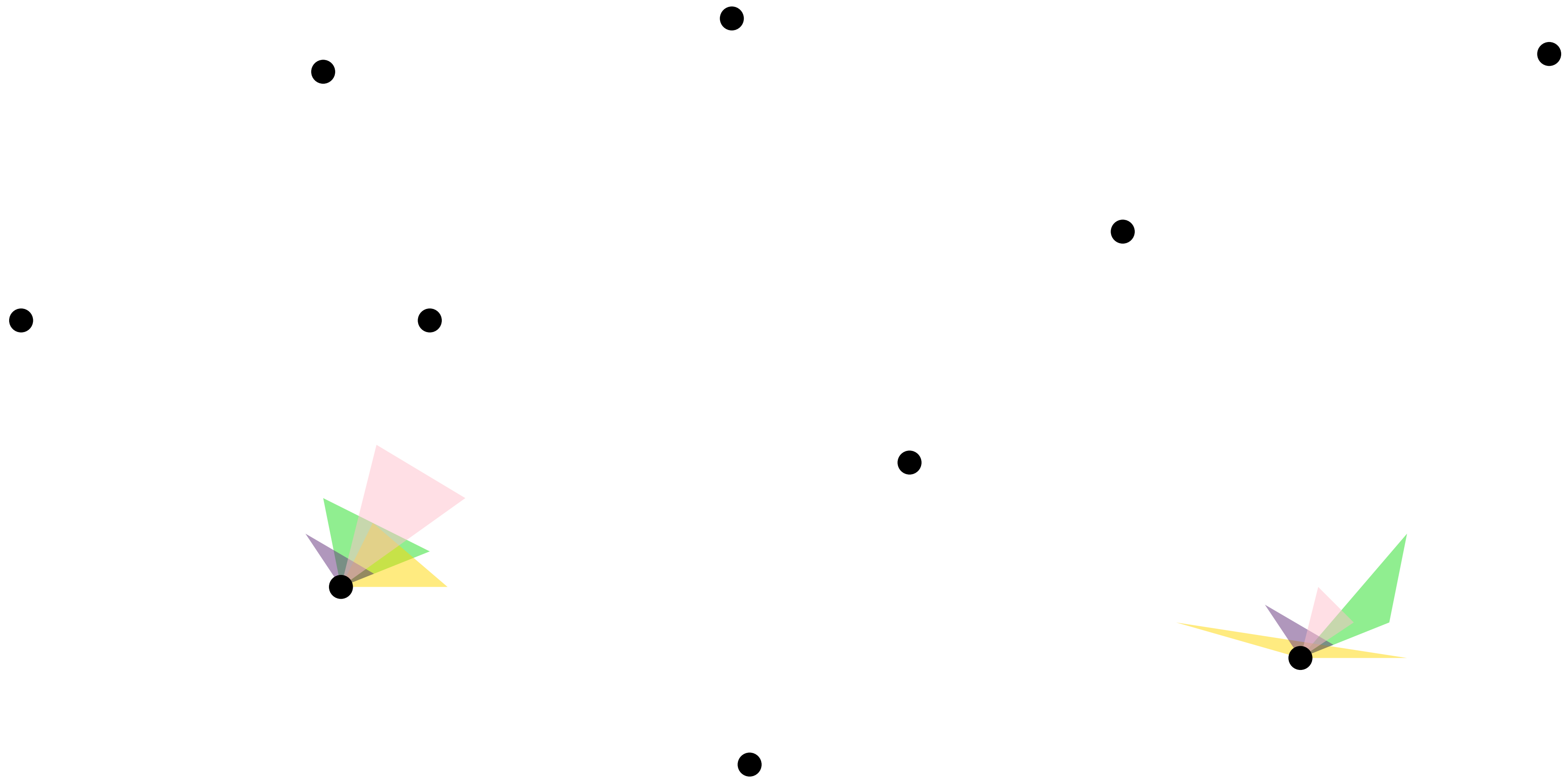
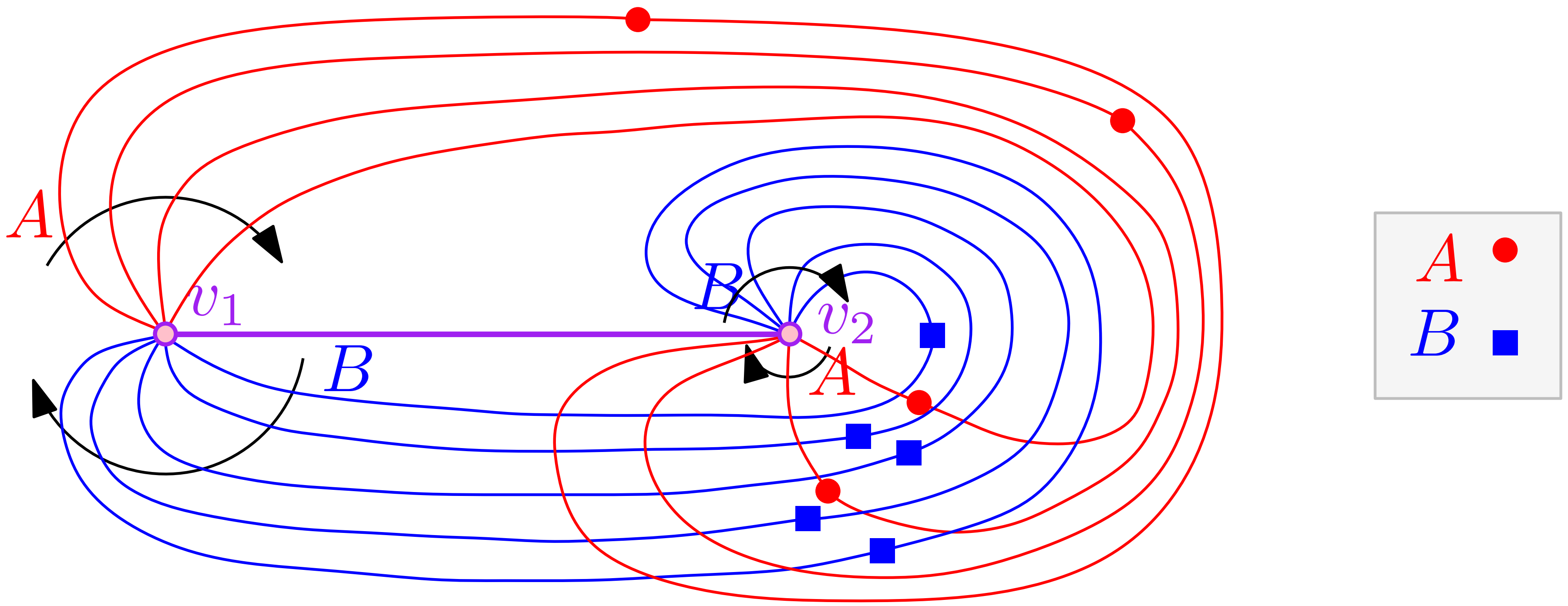


Characterization via Pair with Bipartition: Glimpse at Proof Sketch

A realizable rotation system of K_n with $n \geq 3$ is generalized twisted
$\Leftrightarrow$ there exist two vertices v_1 and v_2 and a bipartition $A \cup B$ of the remaining vertices
(A or B can be empty) such that:

- $\forall$ a, a' in A, the edge aa' crosses v_1v_2.
- $\forall$ b, b' in B, the edge bb' crosses v_1v_2.
- $\forall$ $a \in A$, $b \in B$, the edge ab does not cross v_1v_2.
- Rotation of v_1: Beginning at v_2, all vertices in B appear before all vertices in A.
- Rotation of v_2: Beginning at v_1, all vertices in B appear before all vertices in A.



Characterization via Pair with Bipartition: Glimpse at Proof Sketch

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$\Leftrightarrow$ pair of antipodal vi-cells