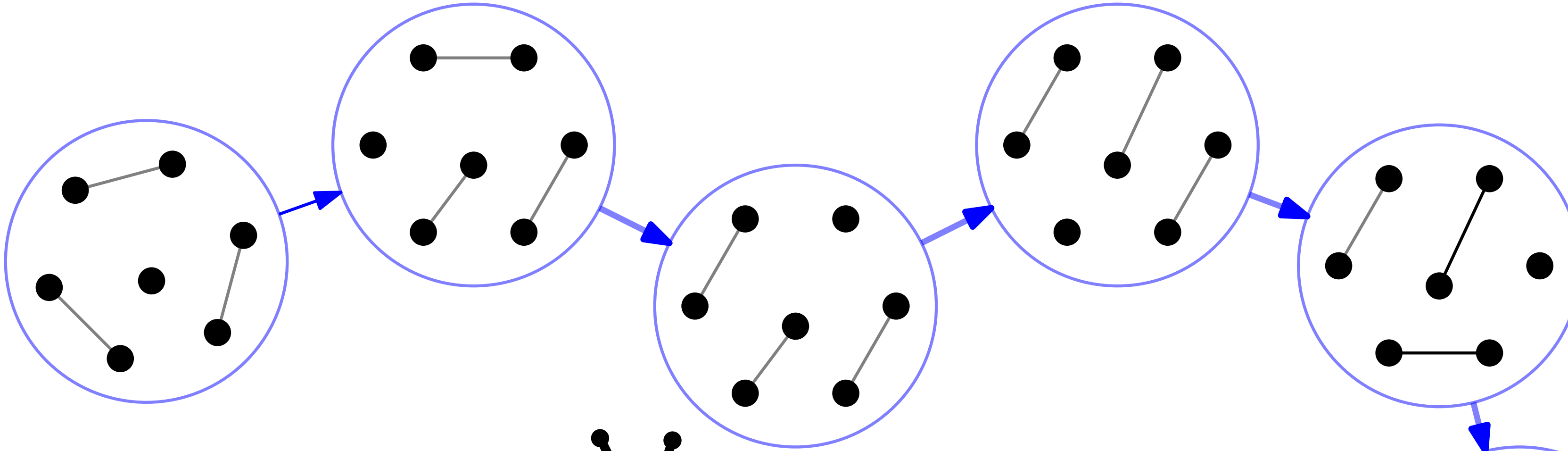
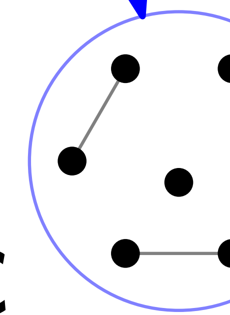


Flipping odd Matchings in geometric and combinatoric matchings



Oswin Aichholzer, Sofia Brenner,
Joseph Dorfer, Hung P. Hoang, **Daniel Perz**,
Christian Rieck, Francesco Verciani

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Italiadomani PIANO NAZIONALE DI RIPRESA E RESILIENZA
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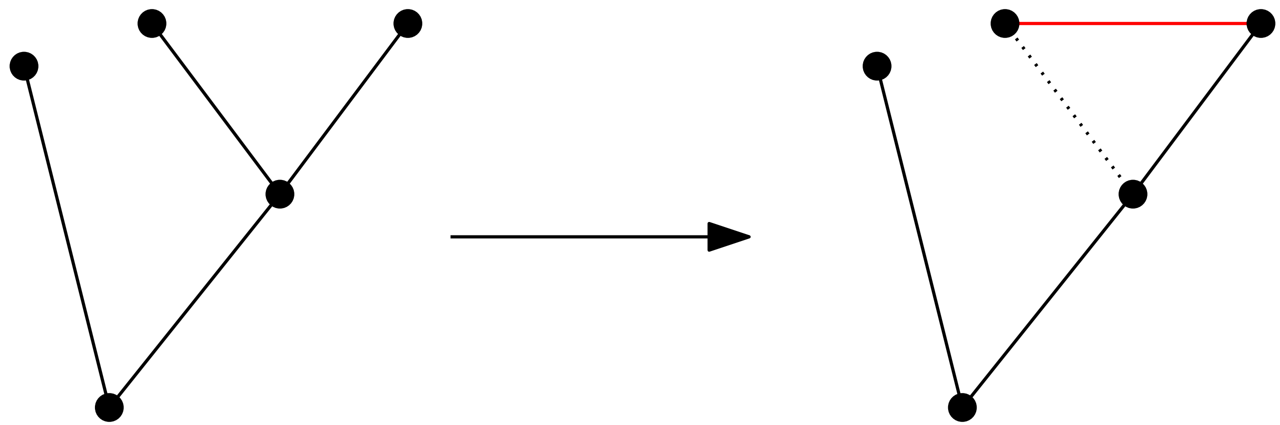
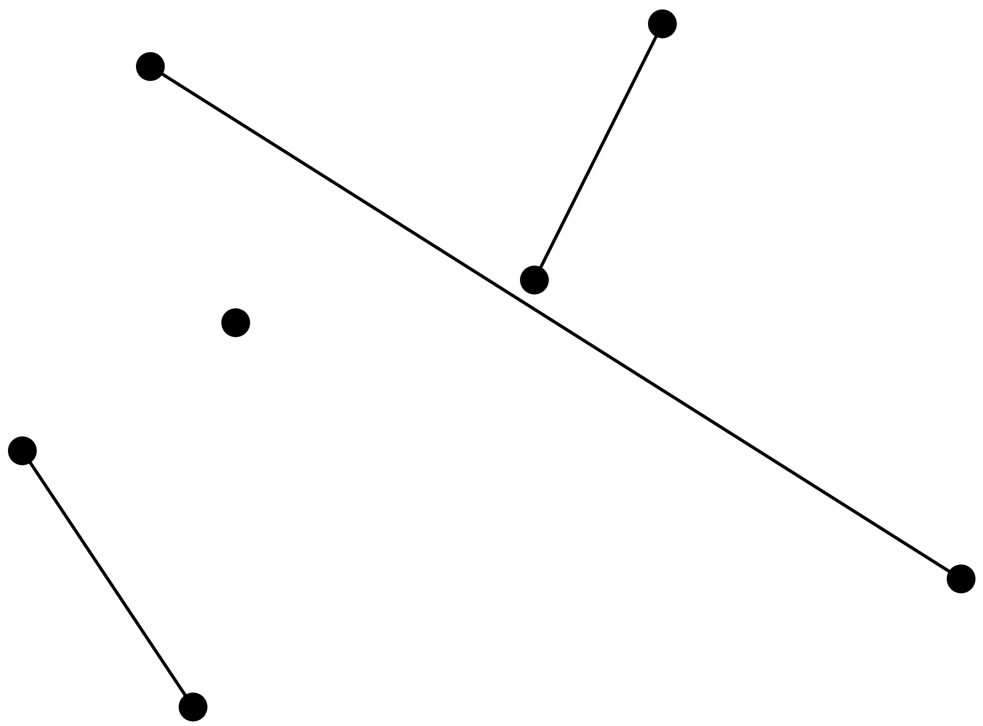
Flipping ...

Edge flip: Replace one edge with another (allowed) edge.



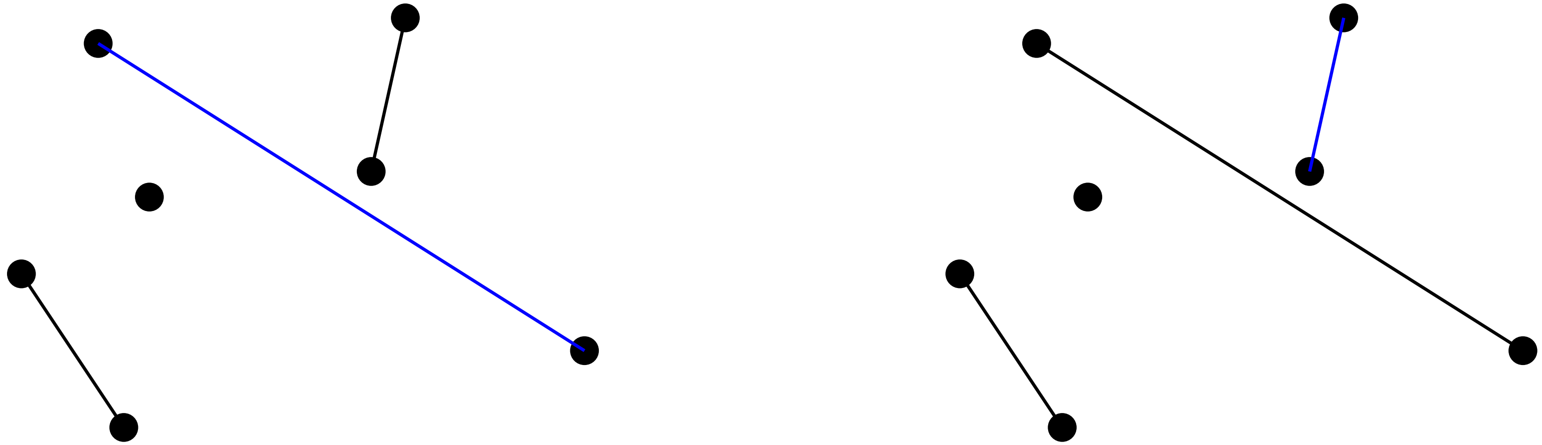
... odd matchings ...

Set with $2m + 1$ vertices.
Matching with m edges, one vertex is isolated



… in geometric and combinatorial settings

Geometric setting: No crossings allowed!



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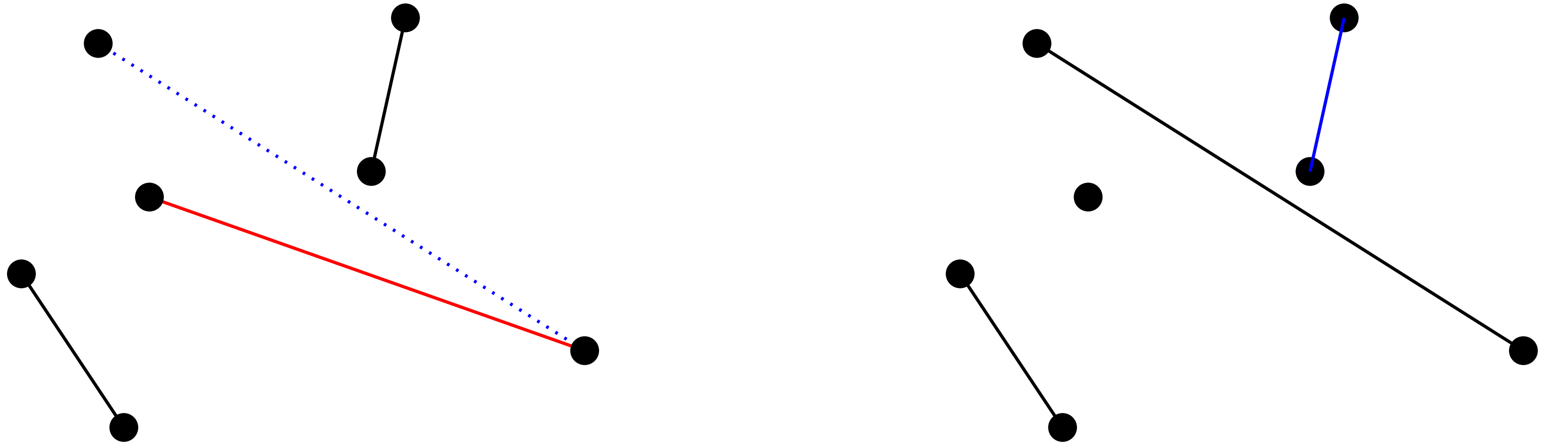

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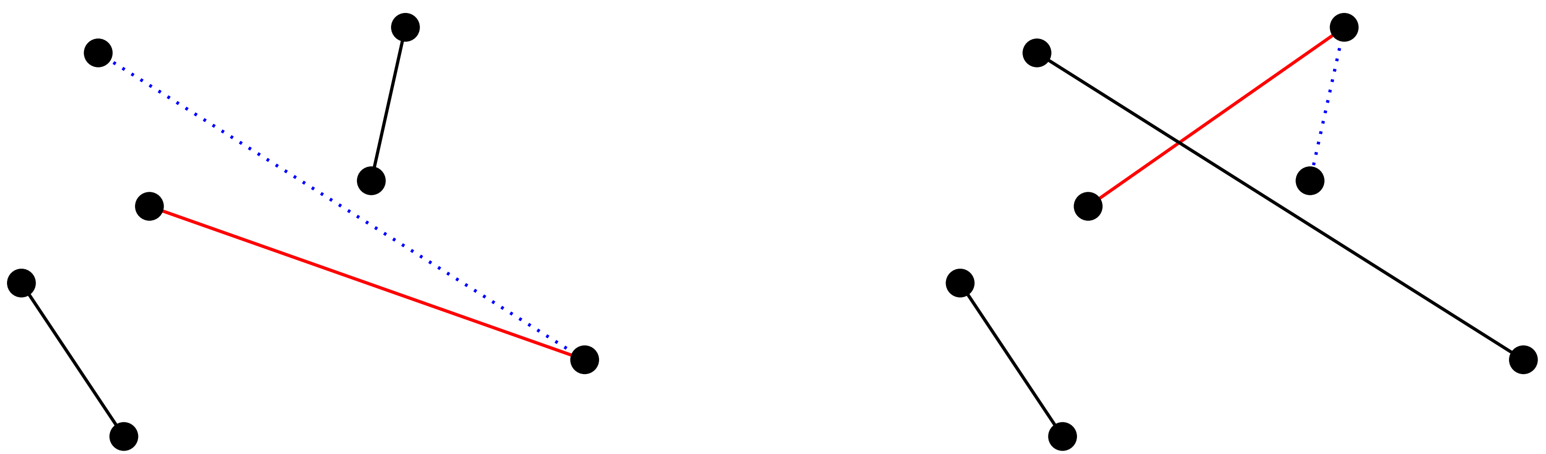

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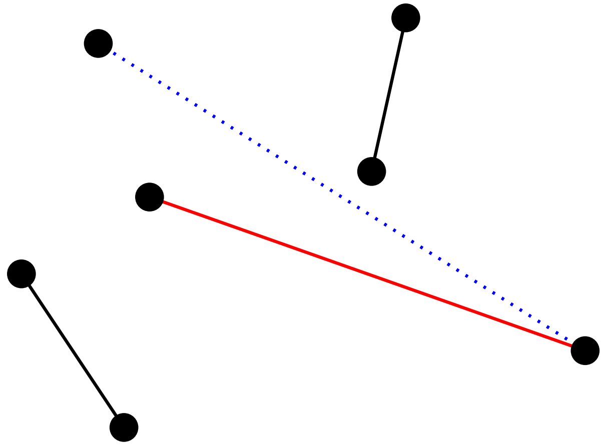
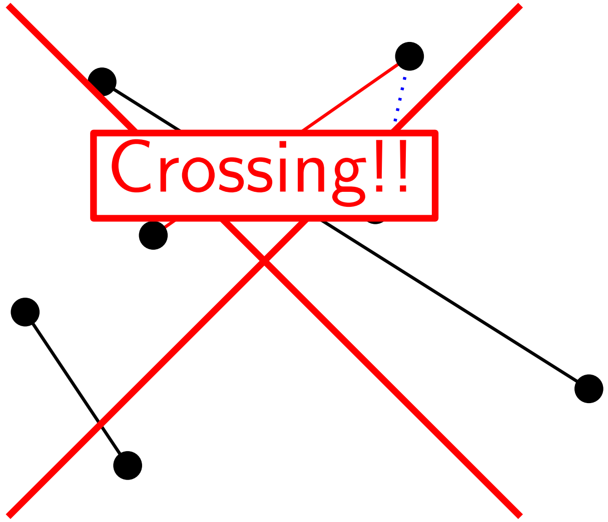
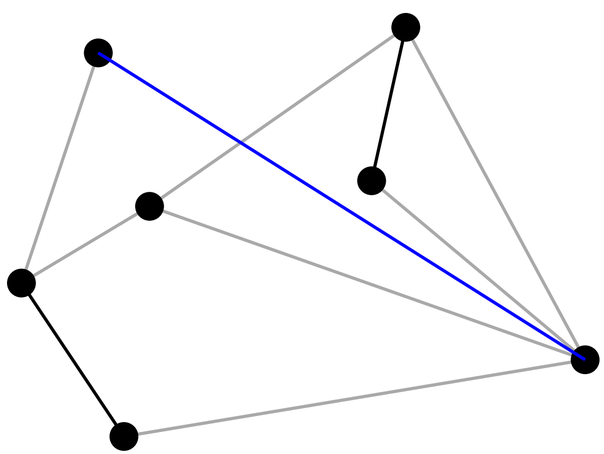


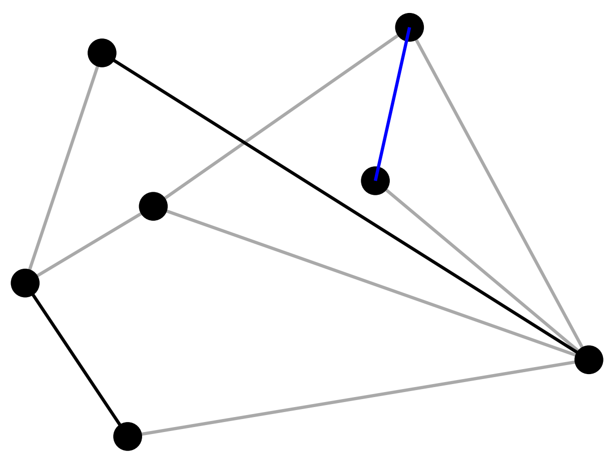


... in geometric and combinatorial settings

Combinatorial setting: Given an underlying graph (in gray).

Matching edges are also edges of the underlying graph.





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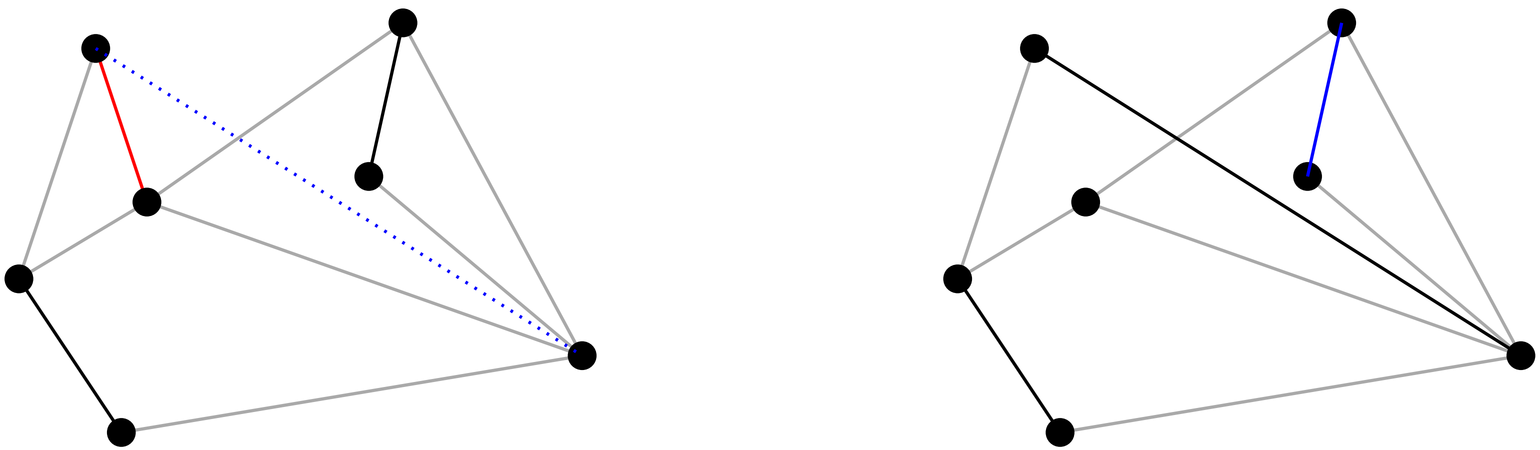

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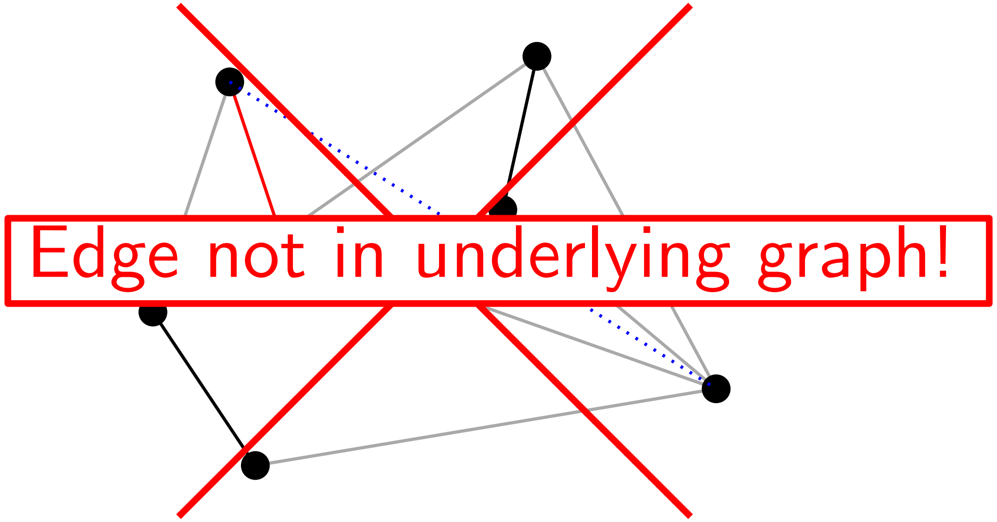

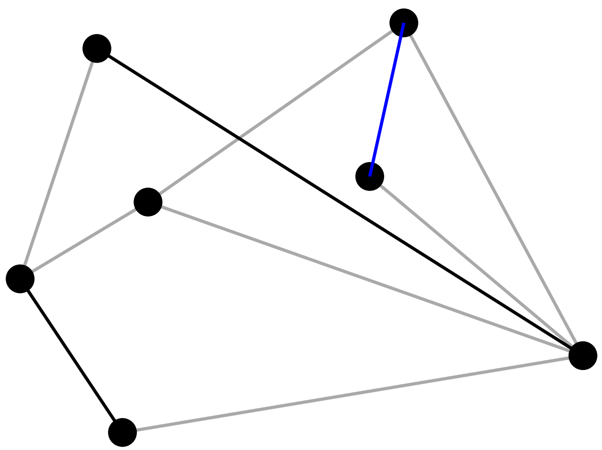

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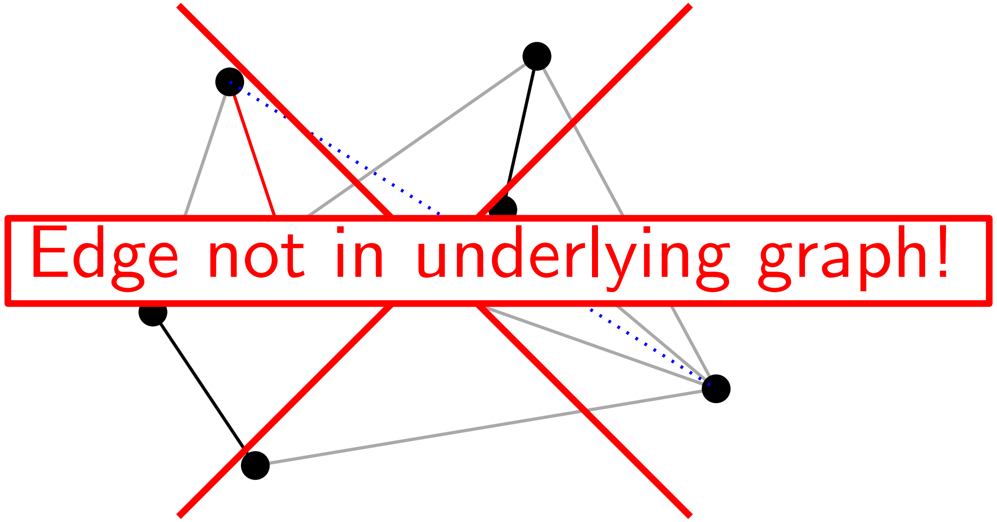

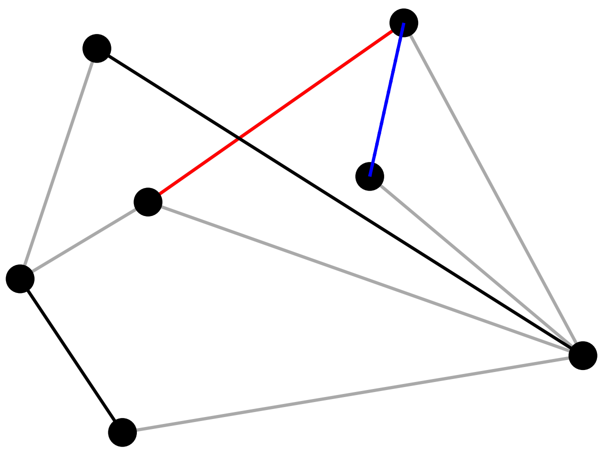

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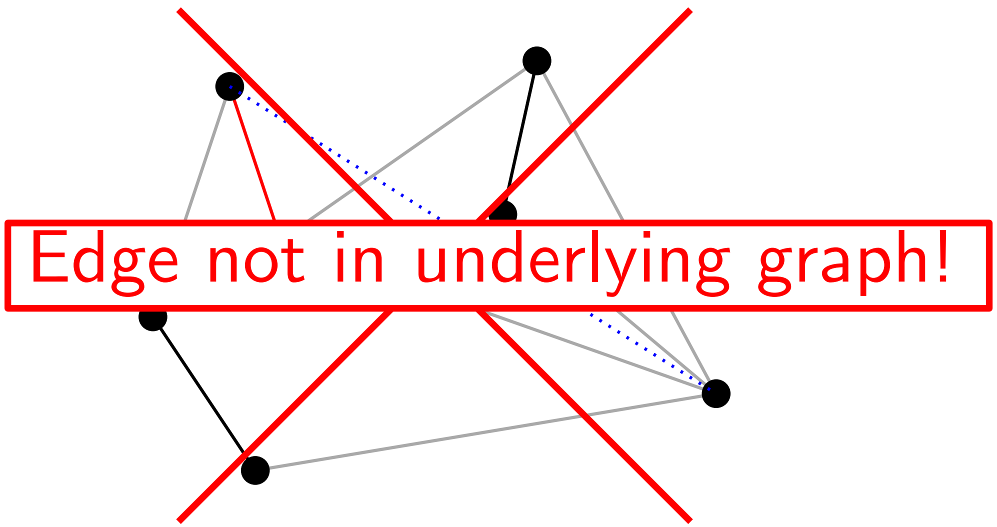
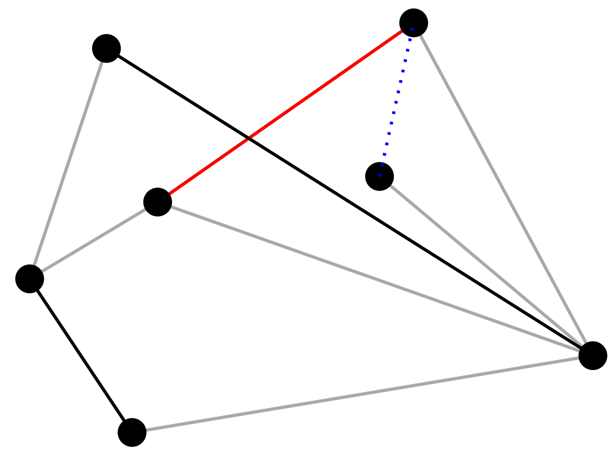
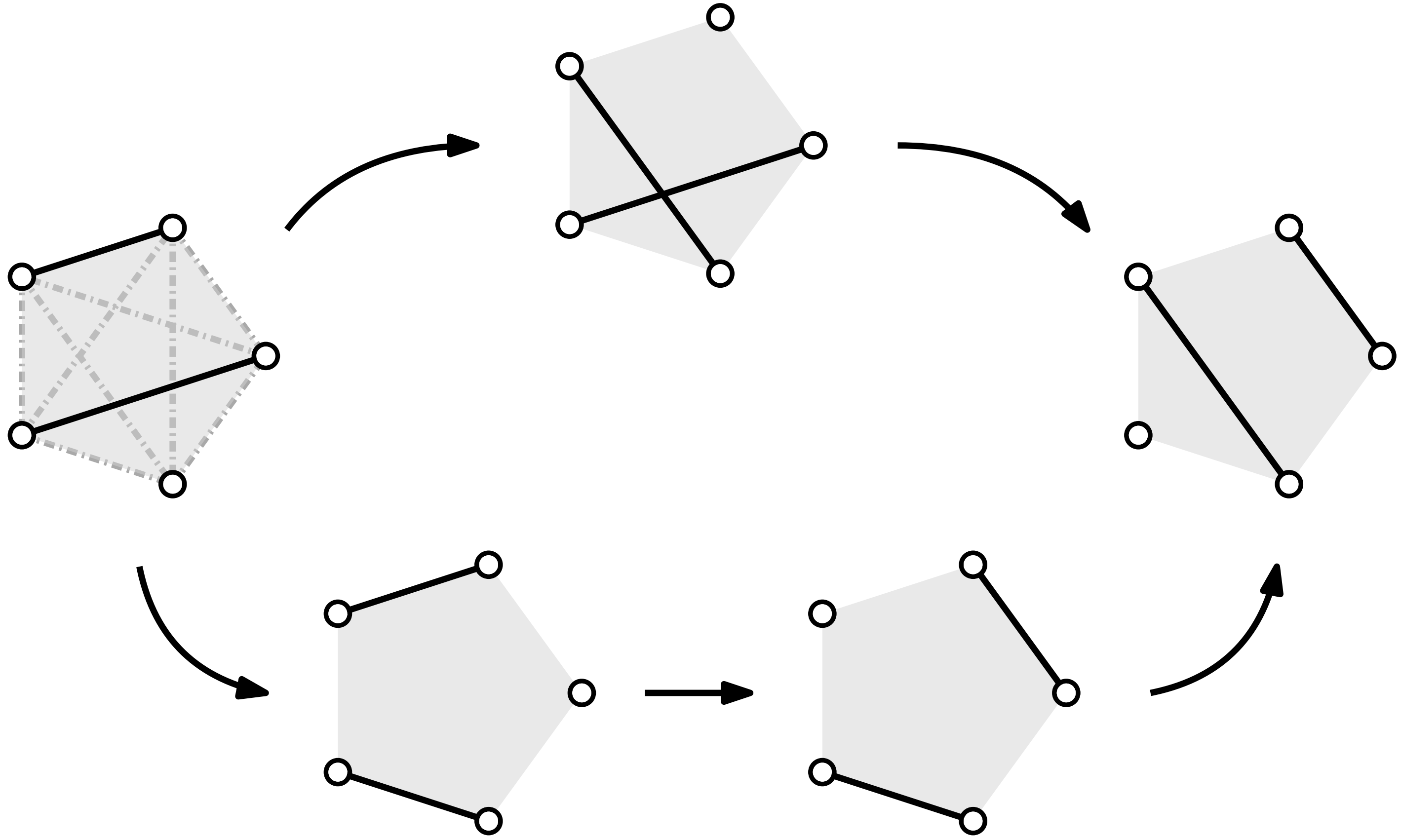




… in geometric and combinatorial settings

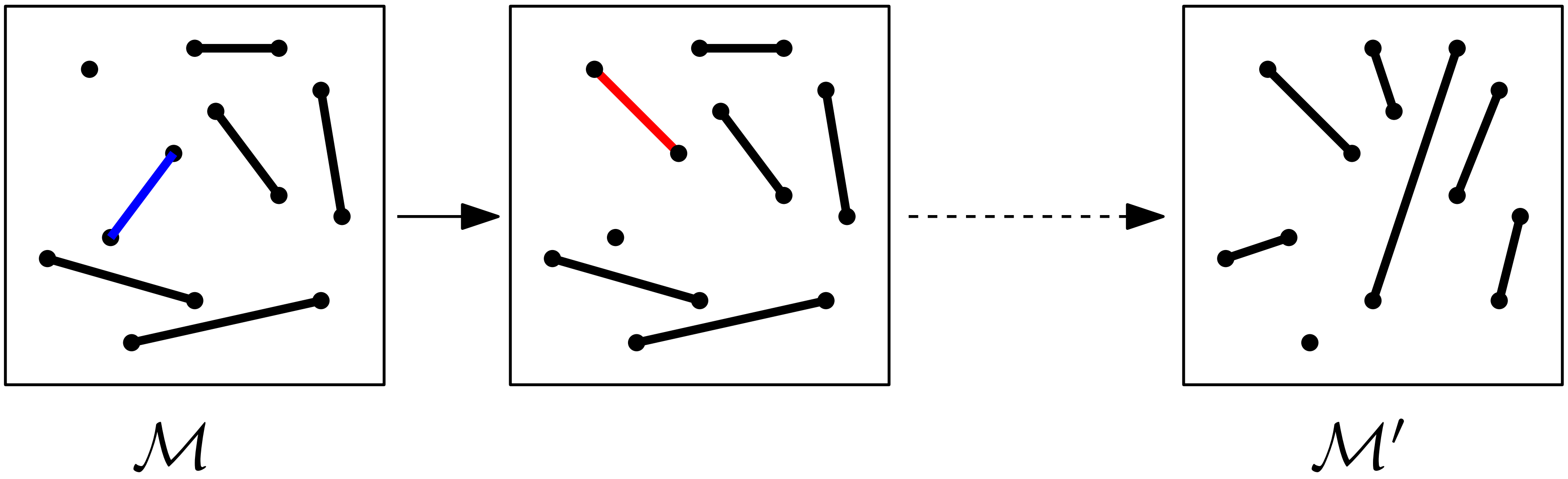
Combinatorial setting



Geometric setting

Classic questions in reconfiguration

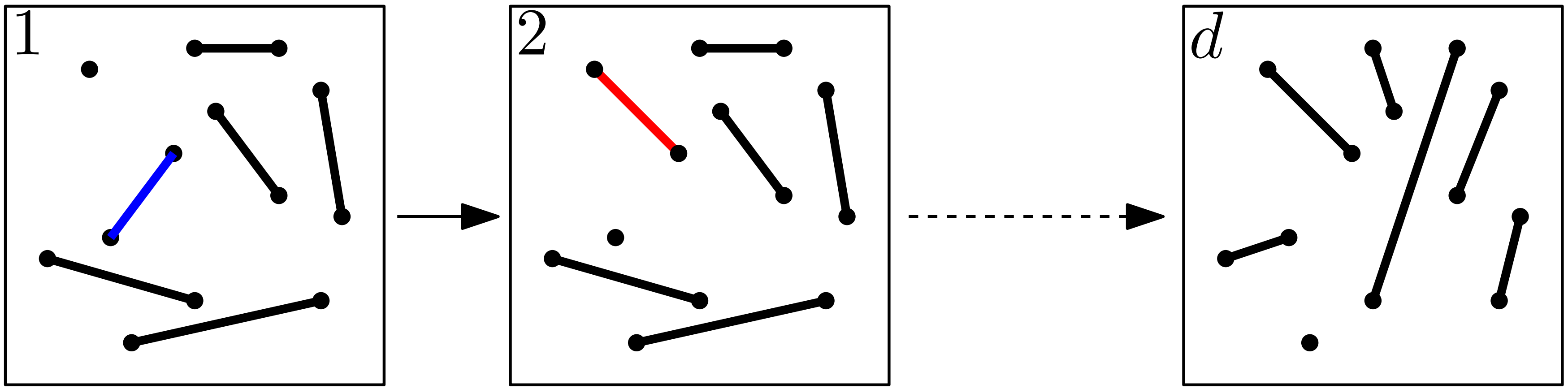
Connectivity: Can any odd matching be transformed into any other odd matching via flips?



Classic questions in reconfiguration

Connectivity: Can any odd matching be transformed into any other odd matching via flips?

Diameter: What is the smallest number d such that any odd matching can be transformed to any other odd matching with d flips?



$\mathcal{M}$ $\mathcal{M}'$

Classic questions in reconfiguration

Connectivity: Can any odd matching be transformed into any other odd matching via flips?

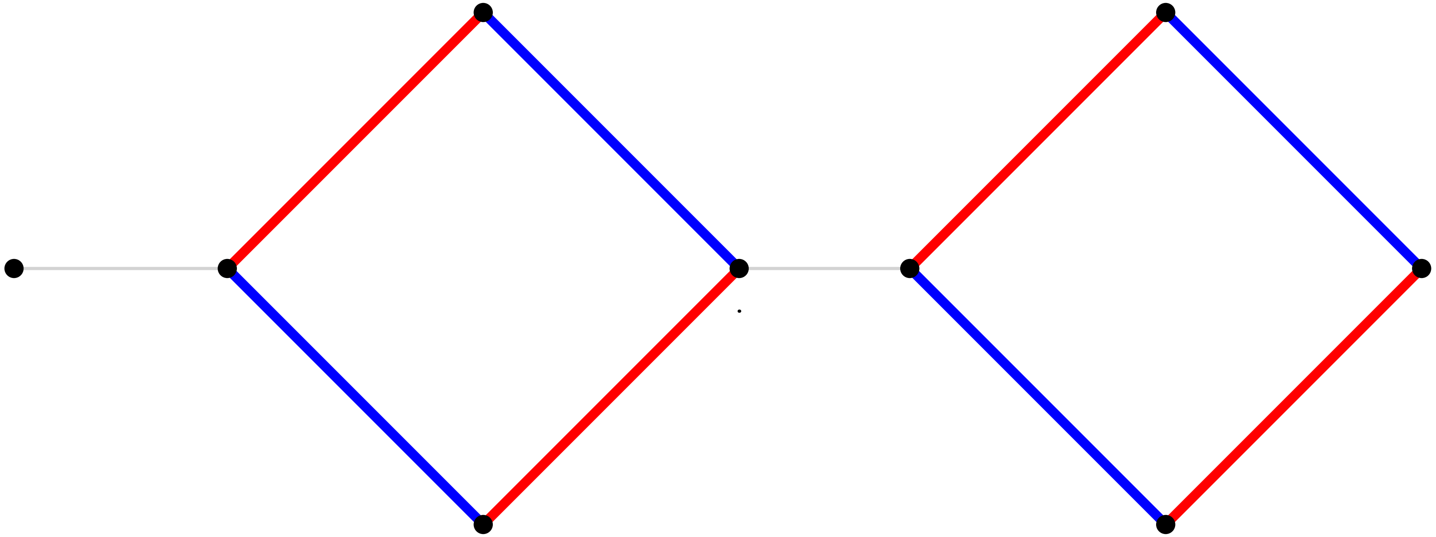
Diameter: What is the smallest number d such that any odd matching can be transformed to any other odd matching with d flips?

Complexity: Given two odd matchings $\mathcal{M}, \mathcal{M}'$ and an integer k. What is the computational complexity to decide whether $\mathcal{M}$ can be transformed into $\mathcal{M}'$ in at most k flips.

Results

Setting	Connect.	Diam.	Compl.
Convex	Yes [ABPS]	$\leq \frac{3n}{2} - \mathcal{O}(1)$	Open
General	Yes [ABPS]	$\mathcal{O}(n^2)$ [ABPS]	NP
Grid	Yes	$\Theta(n)$	NP
Graphs	Charact.	$\Theta(n)$	NP

Combinatorial: Connectedness



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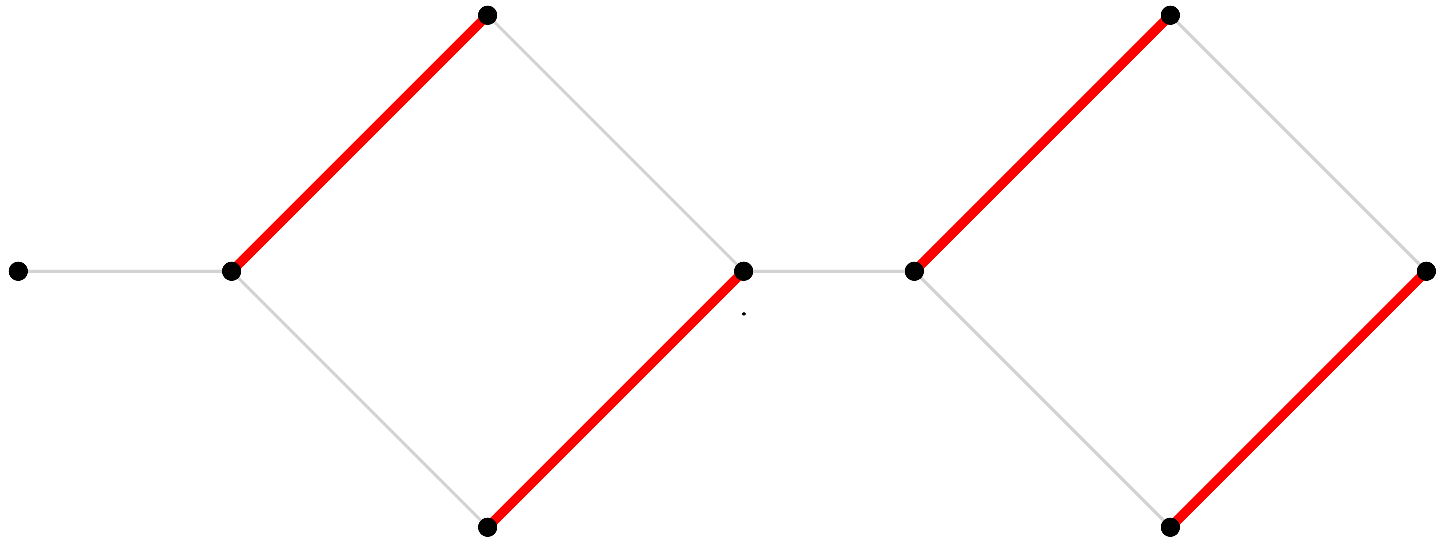

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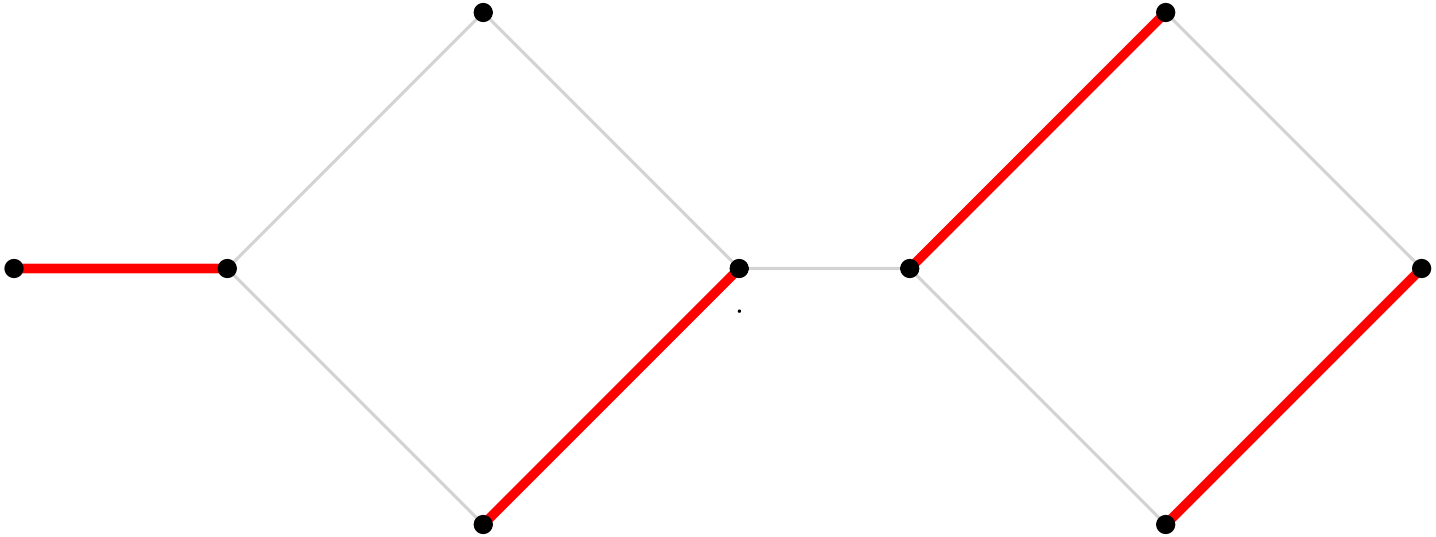

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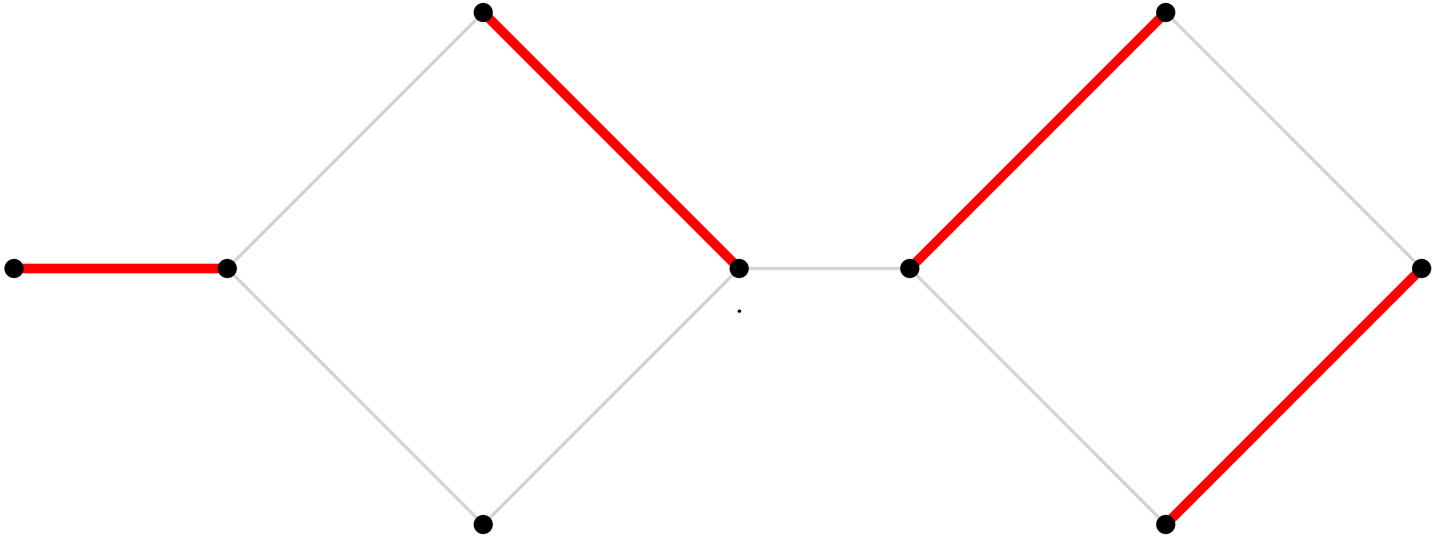

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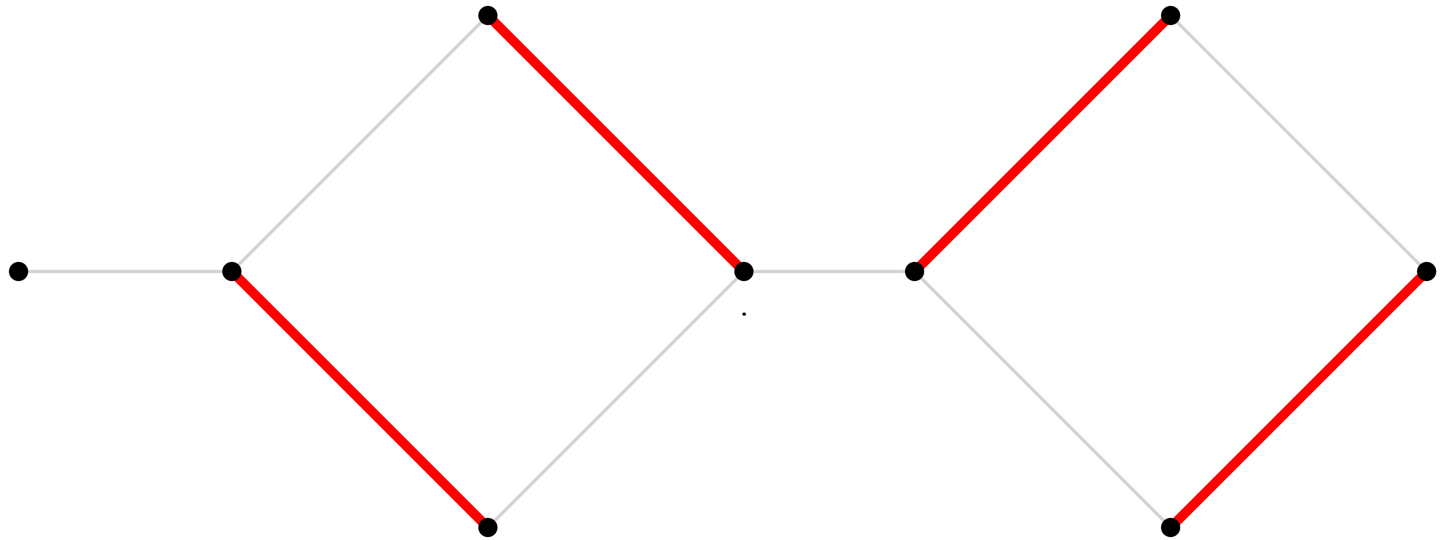
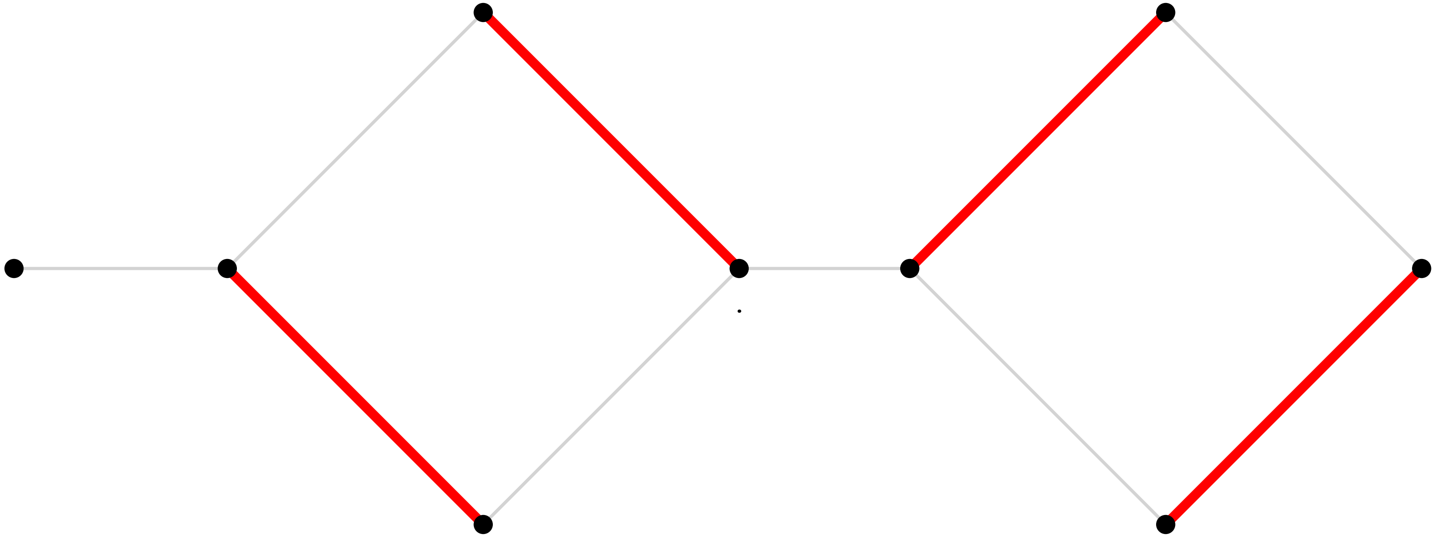


Can't get from M to M' with flips!

Combinatorial: Connectedness



Can't get from M to M' with flips!
For which graphs can we transform any odd matching into any other odd matching via flips?

Characterization for graphs

Observation:
- If an edge is in no odd matching, we can omit it.
- If an edge is in every odd matching we can omit its vertices.

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- If an edge is in no odd matching, we can omit it.
- If an edge is in every odd matching we can omit its vertices.

Theorem: Given a graph G. We can transform any odd matching to any other odd matching on G if and only if for any edge $e = (u, v)$ one of the following holds:
- e is in every or in no odd matching of G.
- $G - u$ or $G - v$ admits a plane perfect matchings.

Characterization algorithm

For every edge $e = (u, v)$:
- Check whether $G - u$ or $G - v$ has a perfect matching.
- or $G - e$ has an odd matching (e in none).
- or $G - u - v$ has an odd matching (e in every).

If this is true for every edge, then connected

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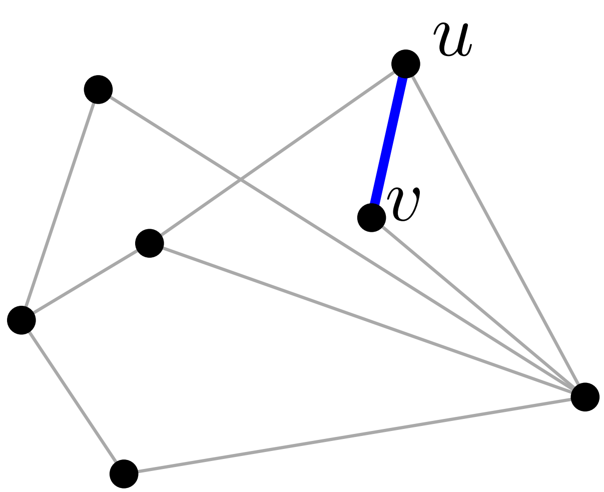

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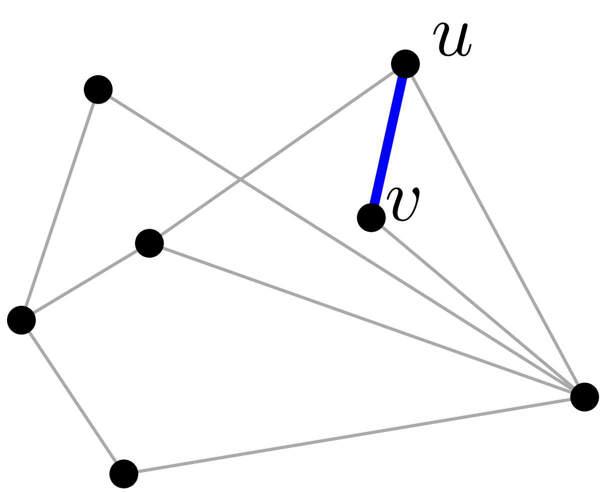

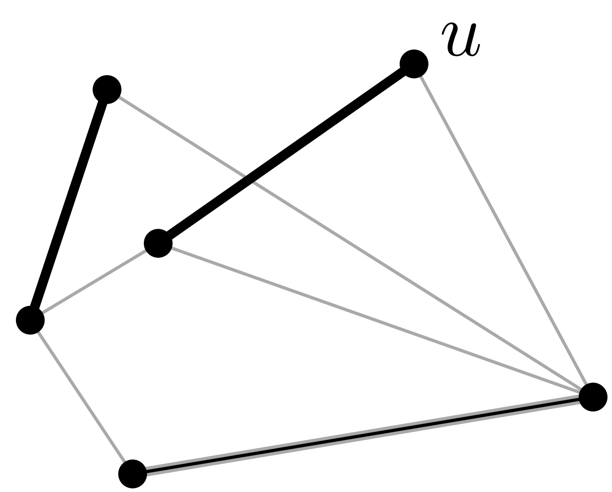

$G - v$ has a perfect matching!

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Checking if a graph has a perfect or odd matching:
- $\mathcal{O}(n^{2.5})$. [Micali & Vazirani 1980]

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Checking if a graph has a perfect or odd matching:
- $\mathcal{O}(n^{2.5})$. [Micali & Vazirani 1980]
- $\mathcal{O}(n^2)$ edges to check.

Total runtime: $\mathcal{O}(n^{4.5})$

Convex point sets: definitions

Happy edge: An edge that is both in M and M'.
Bad happy edge: A happy edge that splits the point set such that the isolated vertex is in one part and M differs from M' in the other part.

M $\qquad\qquad$ M'

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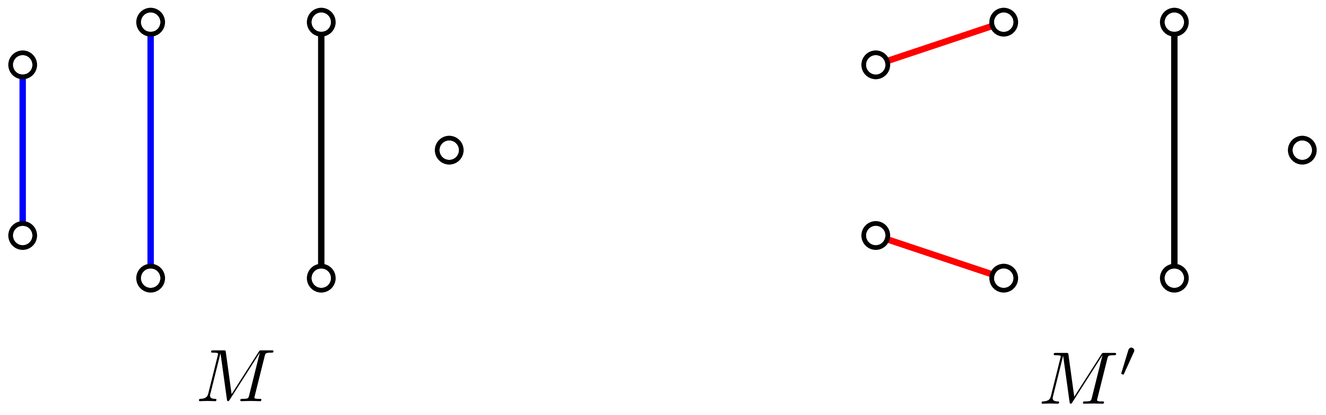

Note: Happy edges on the convex hull are never bad!

Cute little lemma

Lemma:

Given two odd matchings M and M', such that $M \cup M'$ is crossing free, and let

-B be the number of bad happy edges of $M \cup M'$,

-C be the number of edges of M in cycles of $M \cup M'$,

-c be the number of cycles of $M \cup M'$,

-D be the edges of M in a path of $M \cup M'$.

Then transforming M into M' needs $2B + C + 2c + D$ flips.

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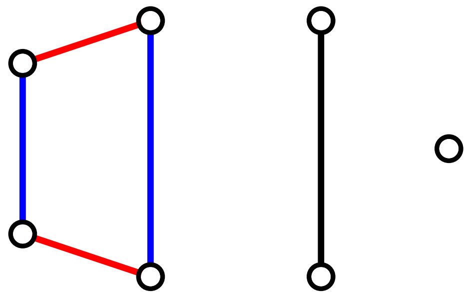

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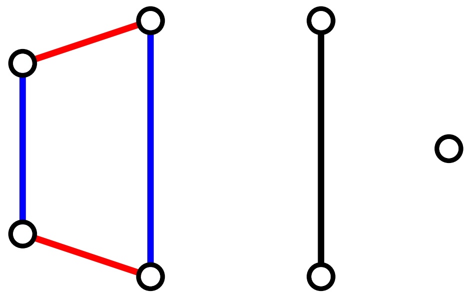

$B = 1,\ C = 2,\ c = 1,\ D = 0$

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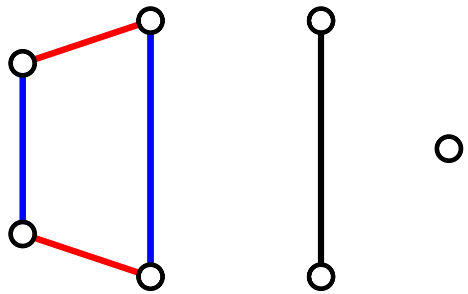
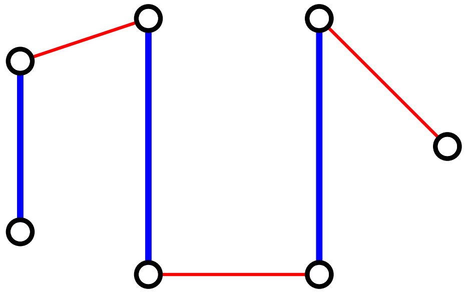


$B = 1$, $C = 2$, $c = 1$, $D = 0$

6 flips.

Upper bound of convex set

Given an odd matching M.
Take any odd matching M_H on the convex hull:
$B = 0$, $c \leq m/2$ cycles, and $C + D \leq m$.



M

At most $3m/2$ flips from M to M_H.

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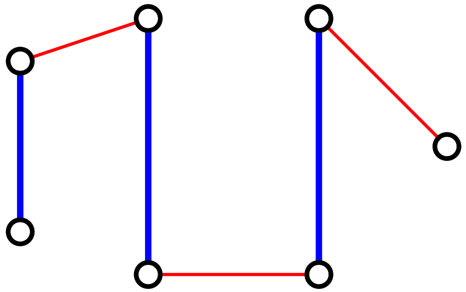

M

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At most $3m = 3n/2$ flips from M to M'.

NP-hardness for geometric

Given a point set and two odd matchings M and M'. How hard is it to decide, whether we can reconfigure M to M' in k flips?

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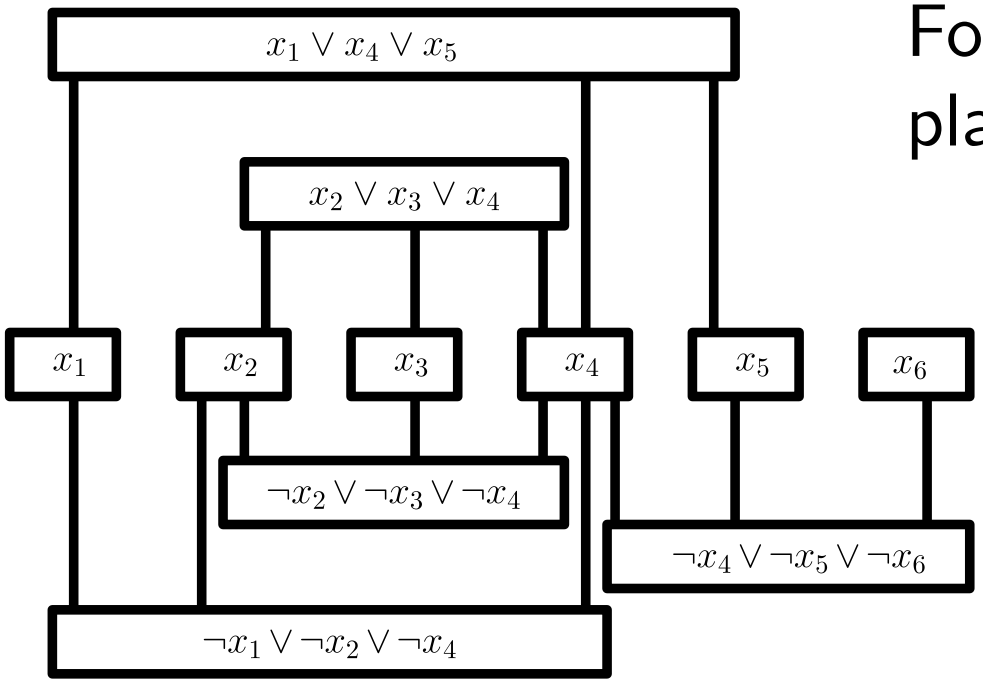
Reduction from PlanarMonotone3SAT

NP-hardness for geometric

Reduction from PlanarMonotone3SAT

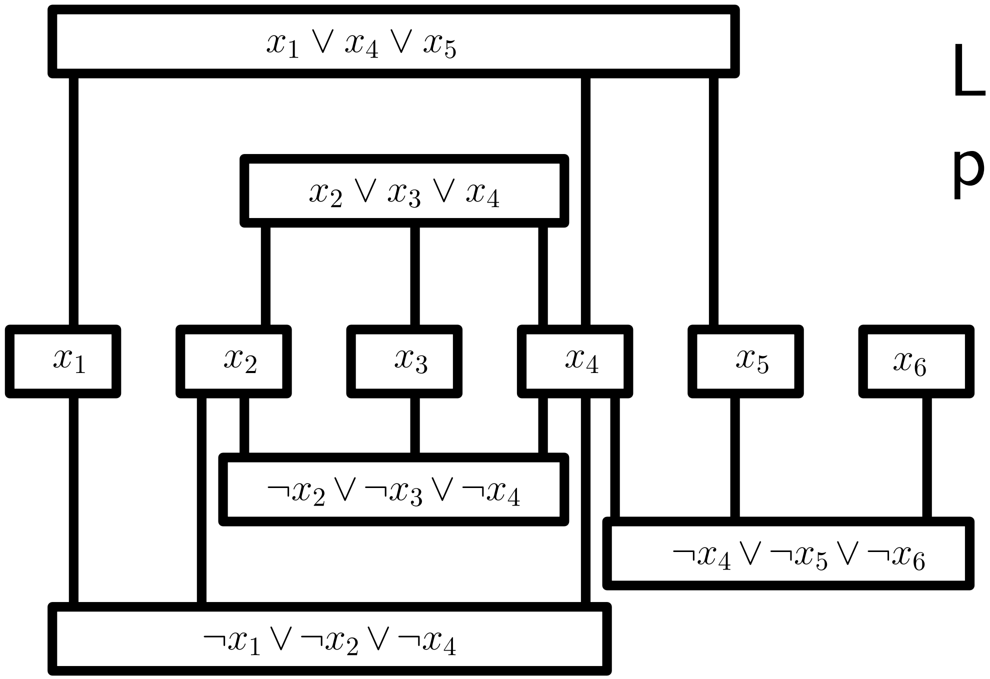


Formula can be drawn as planar incidence graph.

PlanarMonotone3SAT representation of
$(x_1 \lor x_4 \lor x_5) \land (x_2 \lor x_3 \lor x_4) \land (\neg x_1 \lor \neg x_2 \lor \neg x_4) \land$
$(\neg x_2 \lor \neg x_3 \lor \neg x_4) \land (\neg x_4 \lor \neg x_5 \lor \neg x_6)$

NP-hardness for geometric

Reduction from PlanarMonotone3SAT



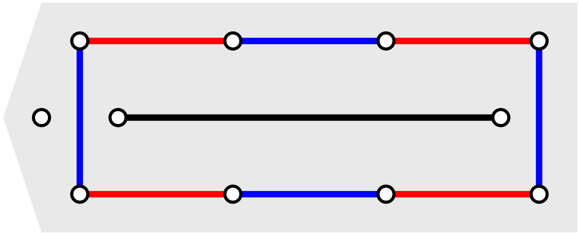
Literals are either all positive or all negative

Variables along a line

PlanarMonotone3SAT representation of
$(x_1 \vee x_4 \vee x_5) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_4) \wedge (\neg x_2 \vee \neg x_3 \vee \neg x_4) \wedge (\neg x_4 \vee \neg x_5 \vee \neg x_6)$

Gadgets

Variable gadget



Gadgets

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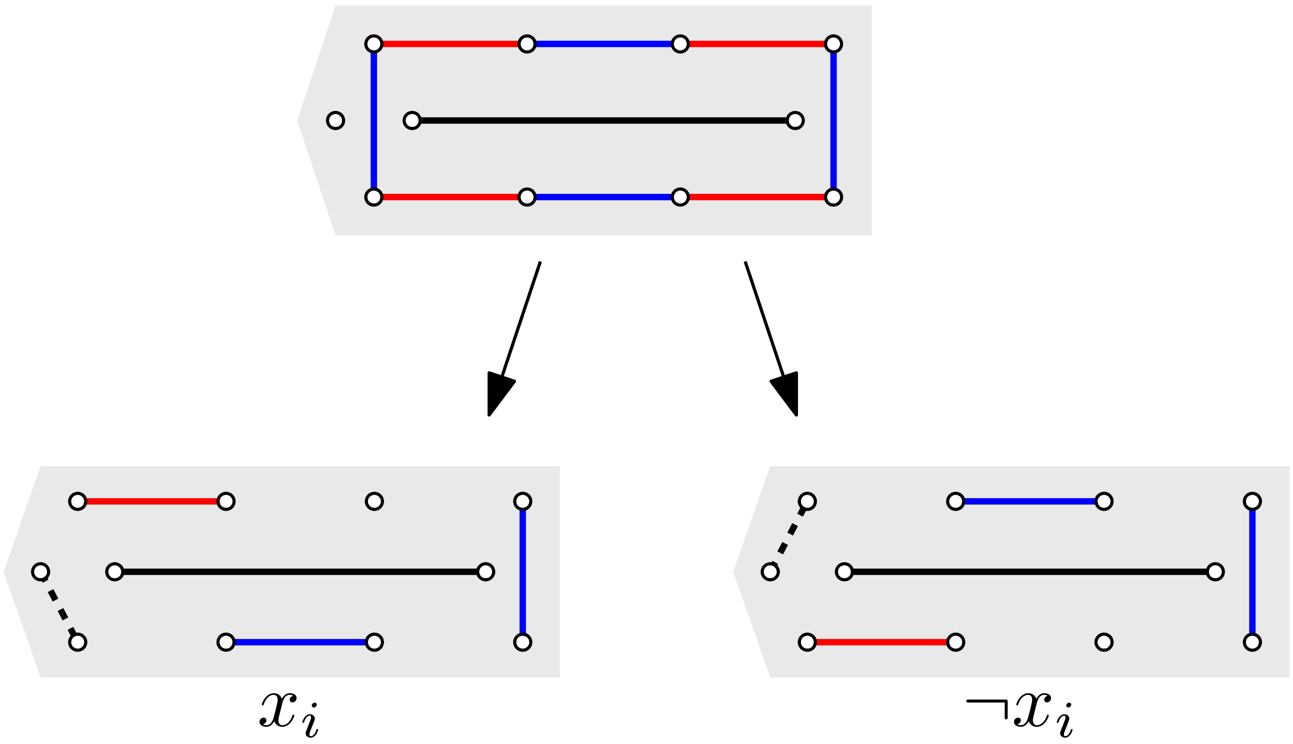
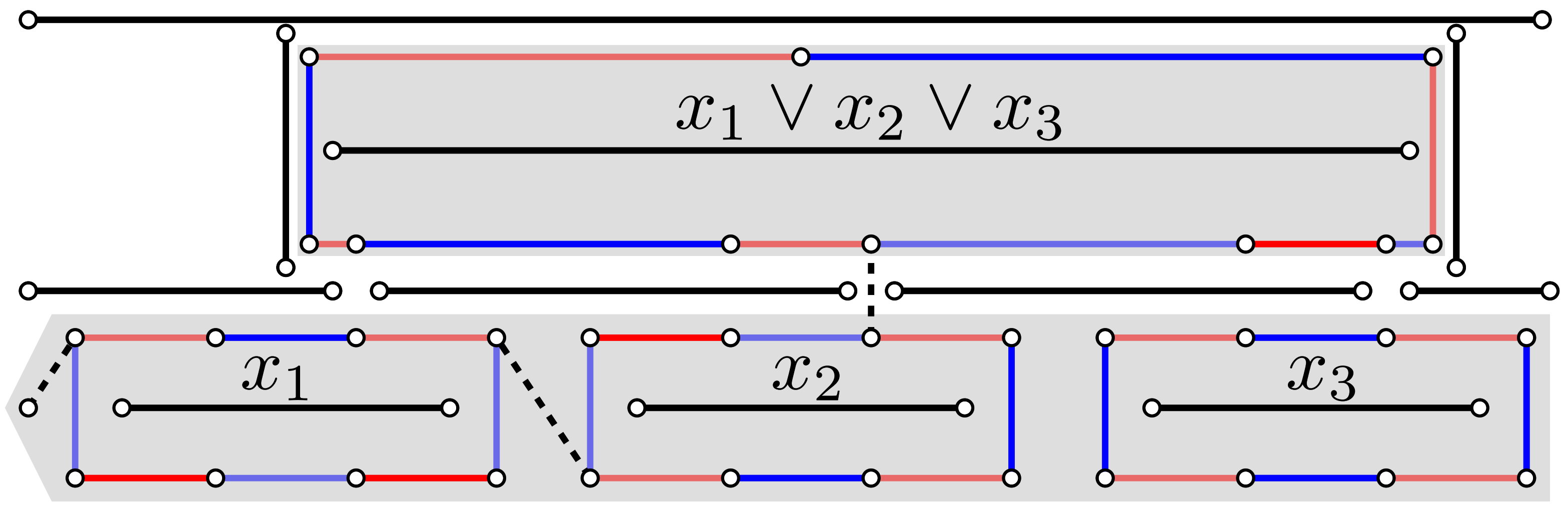
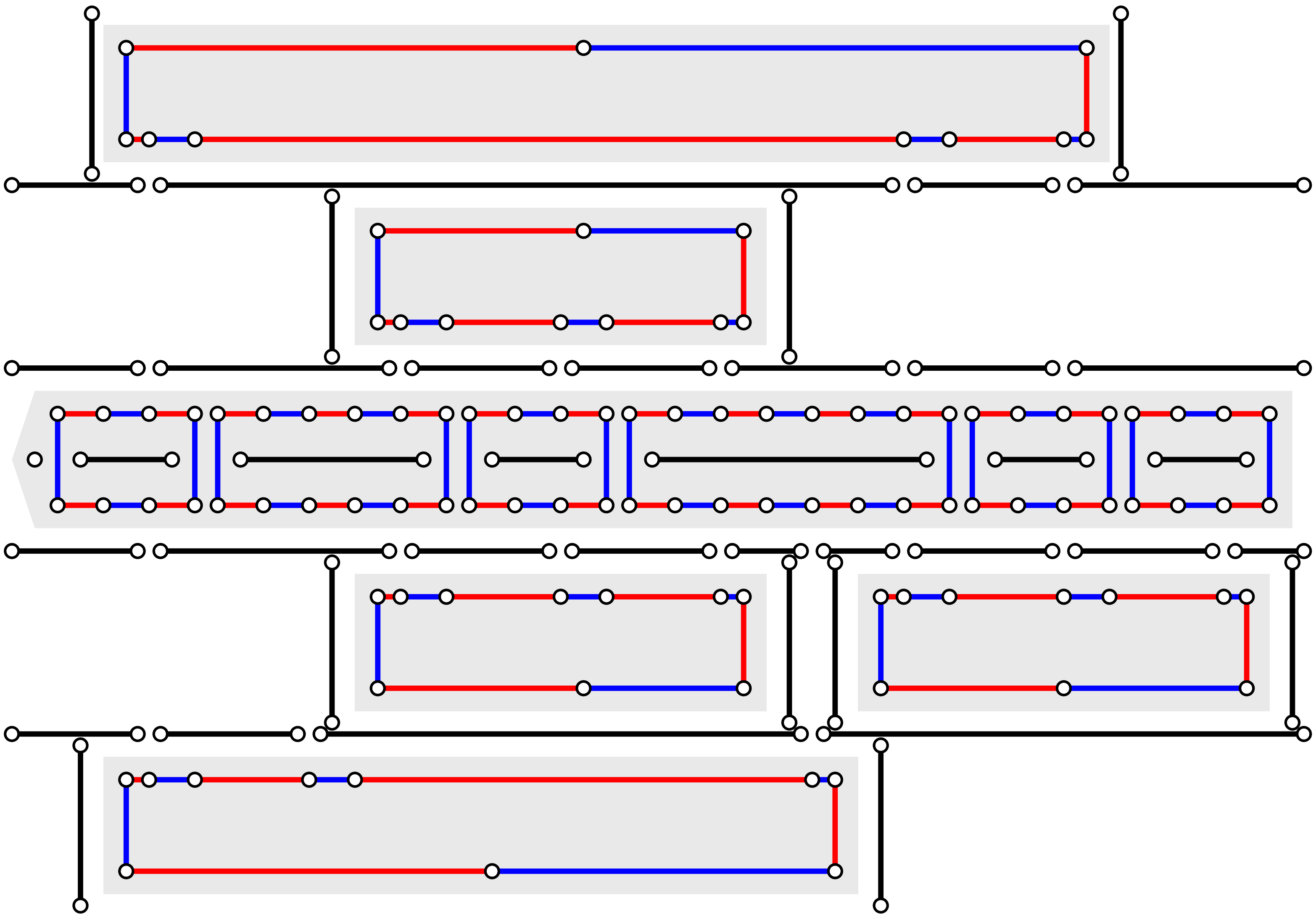


Gadgets

Clause gadget



NP-hardness: Construction



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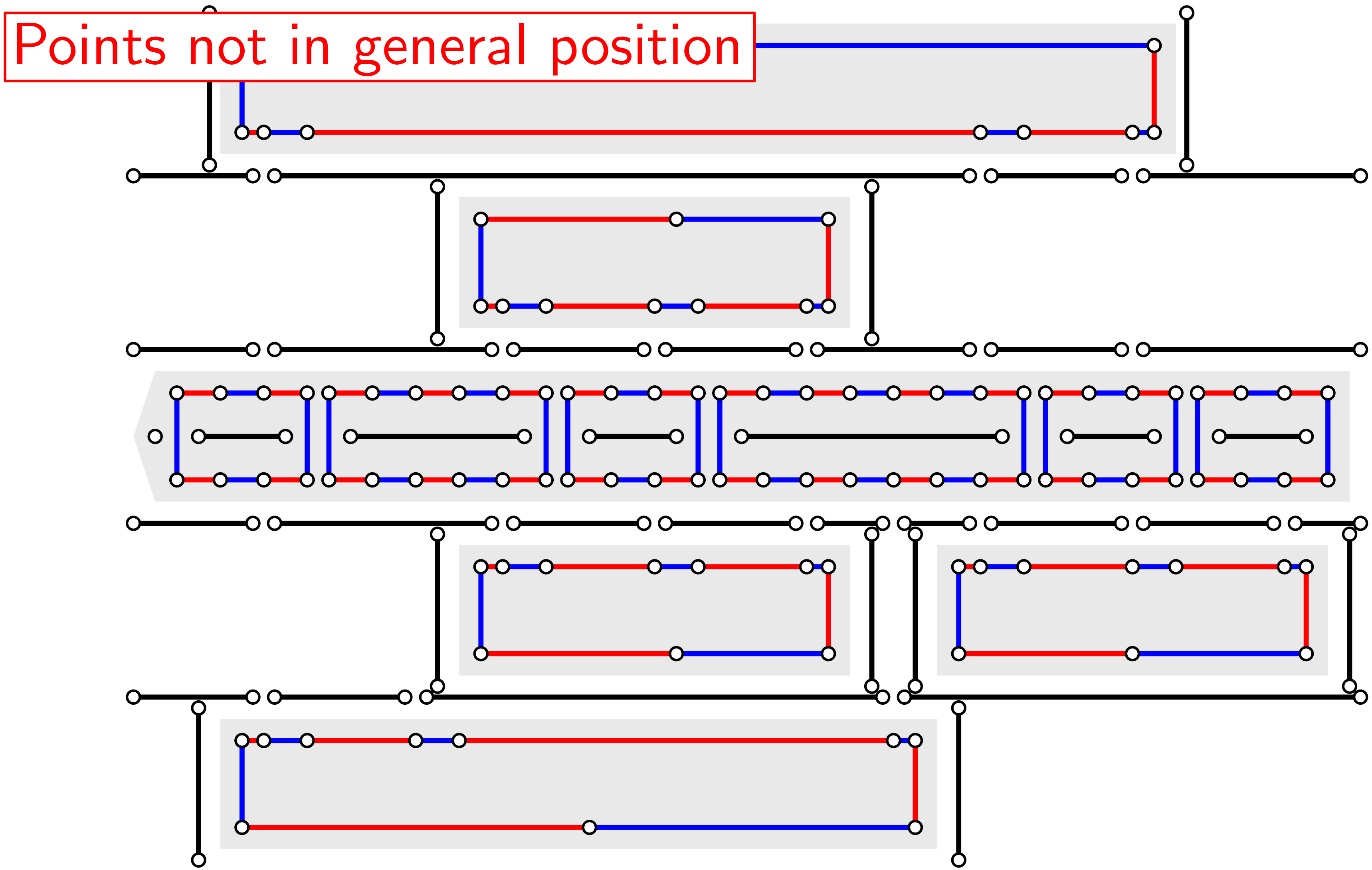

Handling multiple colinear points

Given a point set S of size n on a $w \times h$ grid.
Task: Find mapping ϕ for all points of S, such that
- if a point p is above the line qr,
  then $\phi(p)$ is above the line $\phi(q)\phi(r)$.
- $\phi(S)$ is in general position.
- $\phi(S)$ is on a $f(w) \times g(w)$ grid,
  f, g are polynomial functions.

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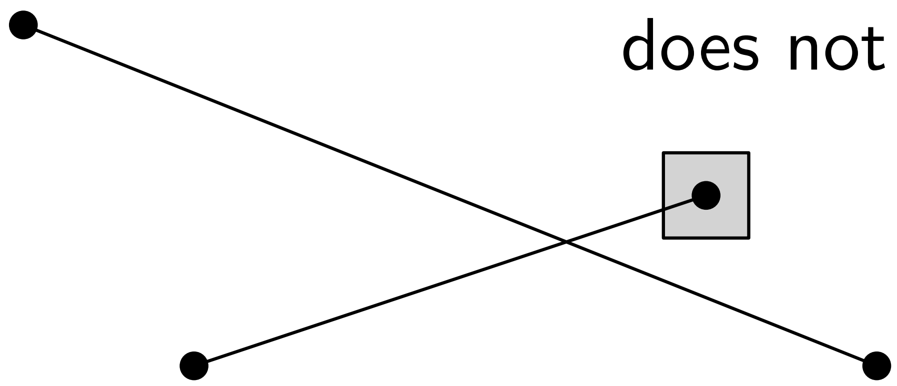
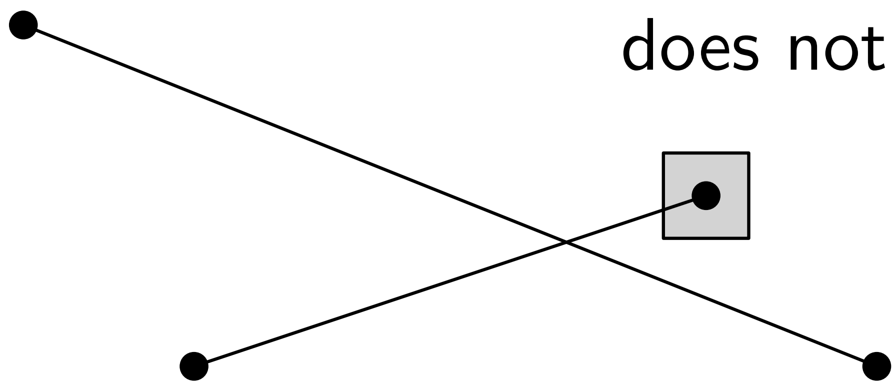
Idea: Place $\frac{1}{16h^3w^3}$ box around each point.
$(n^4 + n) \times (n^4 + n)$ grid inside each box.

Handling multiple colinear points

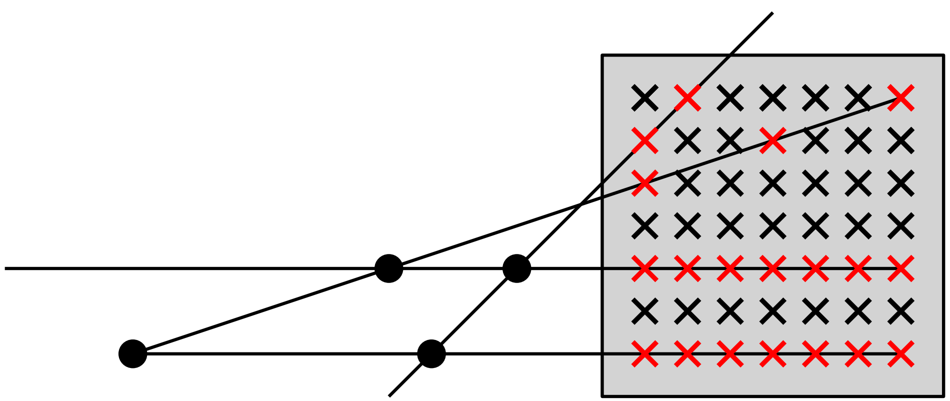


Any line that does not contain p, does not cross the box around p.

Handling multiple colinear points



Any line that does not contain p, does not cross the box around p.



Can choose one point inside box that is not on a line.

Conclusion

Setting	**Connect.**	**Diam.**	**Compl.**
Convex	Yes [ABPS]	$\leq \frac{3n}{2} - \mathcal{O}(1)$	Open
General	Yes [ABPS]	$\Omega(n)$, $\mathcal{O}(n^2)$ [ABPS]	NP
Grid	Yes	$\Theta(n)$	NP
Graphs	Charact.	$\Theta(n)$	NP

Geometric: Improve the bounds
Combinatoric: Which other graph families connected?

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Thank you for your attention