1-planar unit distance graphs with more edges than matchstick graphs

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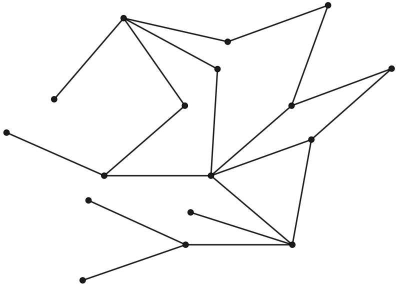
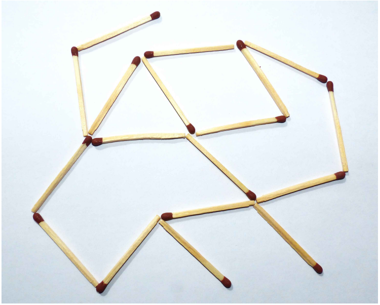
Introduction

- **Def (Harborth, 1981):** A matchstick graph is a plane graph that allows plane embedding with straight edges of equal length.





Introduction

Theorem (Harborth - special case, 1974)

If the unit distance is also the smallest distance among the vertices, then the maximum number of edges for a matchstick graph on n vertices is $\lfloor 3n - \sqrt{12n-3} \rfloor$.

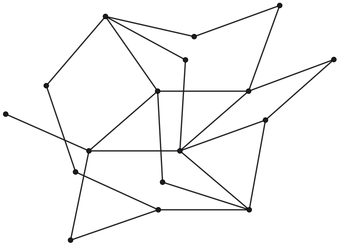
Conjecture (Harborth, 1981)

The maximum number of edges for a matchstick graph on n vertices is $\lfloor 3n - \sqrt{12n-3} \rfloor$.

(Proven by Lavollée and Swanepoel (2023)).

Introduction

- **Def (Gehér and Tóth, 2023):** A 1-planar unit distance graph is a graph that allows a drawing in the plane in which all edges are straight-line segments of equal length and every edge crosses at most one other edge.



Introduction

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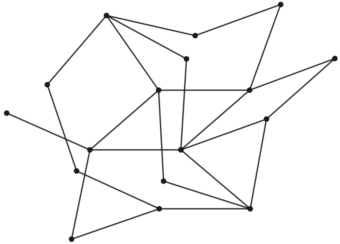
- **Notation:**
  - $u_0(n)$... maximum number of edges of a matchstick graph on n vertices
  - $u_1(n)$... maximum number of edges of a 1-planar unit distance graph on n vertices



What was known

Theorem (Lavollée and Swanepoel, 2023)

For the maximum number of edges of a matchstick graph $u_1(n)$ we have

$$u_0(n) = \left\lfloor 3n - \sqrt{12n - 3} \right\rfloor .$$

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For the maximum number of edges of a 1-planar unit distance graph $u_1(n)$ we have

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Question: (Graph Drawing 2024, Gehér and Tóth)

Is it true that $u_0(n) = u_1(n)$?

Our results

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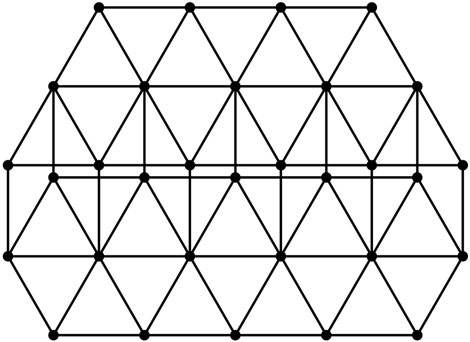

31 vertices, 74 edges ($u_0(31) = 73$)

Our results

Theorem 1

For every $n \geq 16135$, $u_1(n) > u_0(n)$.

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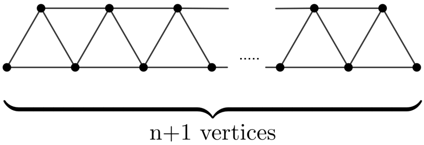
Theorem 2

For any constant $\alpha < \sqrt[4]{\frac{1}{3}} = .7598\ldots$ and for every n sufficiently large (with respect to α), $u_1(n) - u_0(n) \geq \alpha\sqrt[4]{n}$.

The basic construction

Notation:

- Denote by T_n a graph on $2n+1$ vertices and $4n-1$ edges isometric to the set of points $\{[0+\ell, 0], [\frac{1}{2}+m, \frac{\sqrt{3}}{2} : \ell \in \{0, \ldots, n\}, m \in \{0, \ldots, n-1\}]\}$ where two points are connected if their distance is exactly 1. This graph will be called a *path of n triangles*.



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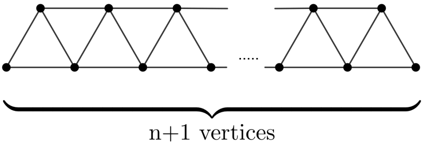
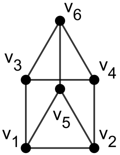
- Denote by F the graph with vertices isomorphic to the points with coordinates $(0,0), (1,0), (0,1), (1,1), (\frac{1}{2}, \frac{\sqrt{3}}{2}), (\frac{1}{2}, 1+\frac{\sqrt{3}}{2})$.





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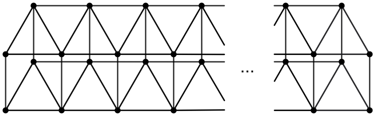

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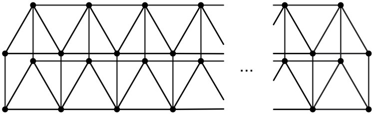

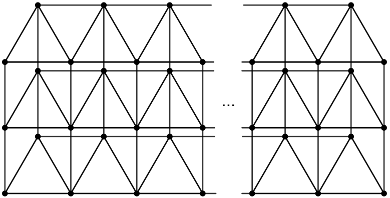

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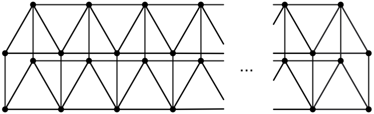

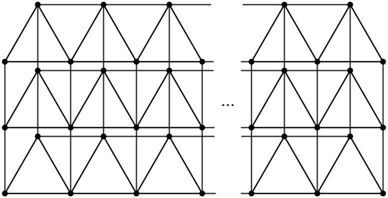

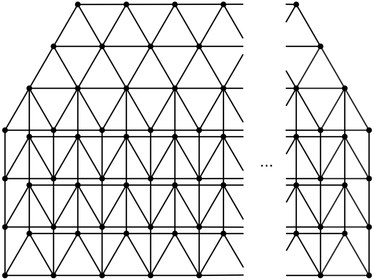

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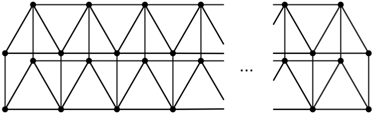

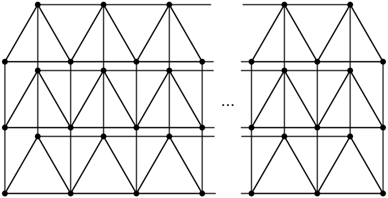

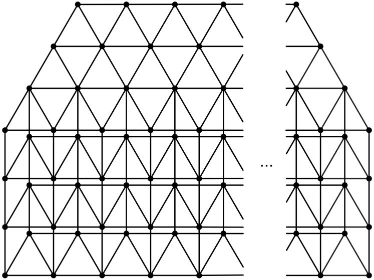

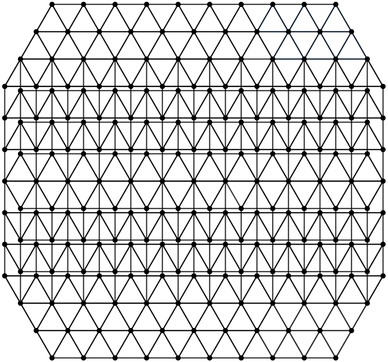

The basic construction

The resulting graph is denoted by $G_{t,k,a}$ with the parameters:

- **k**: the number of $k+1$ copies of the elementary building block F concatenated in a row,
- **t**: the number of such rows stacked on top of each other (to be precise, half of them stacked on top of each other, and the other half added below them by a vertical flip),
- **a**: the number of paths of triangles stacked above (and below) the rows of F's,

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$$|V(G_{t,k,a})| = 2ka - a^2 + a + 4 + 3k + 3t + 2kt,$$

$$|E(G_{t,k,a})| = 6ka - 3a^2 + a + 5 + 7k + 6t + 6kt.$$

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Choose **k=2a+t+1**

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The resulting graph is denoted by $H_{t,a}$ with the parameters:

- **t**: the number of rows of F's stacked on top of each other (to be precise, half of them stacked on top of each other, and the other half added below them by a vertical flip),
- **a**: the number of paths of triangles stacked above (and below) the rows of F's,

$$|V(H_{t,a})| = 3a^2 + 9a + 6at + 7 + 8t + 2t^2,$$

$$|E(H_{t,a})| = 9a^2 + 21a + 18at + 12 + 19t + 6t^2.$$

The basic construction

Proposition

For every even positive integer t and every $a \geq t^2$,
$|E(H_{t,a})| - u_0(|V(H_{t,a})|) \geq t$.

Proof:

For $a = t^2$ it holds $|E(H_{t,t^2})| - u_0(|V(H_{t,t^2})|) > t - 1$.

For every $a > t^2$ denote
$f_t(x) = \sqrt{12(3x^2 + 9x + 6xt + 7 + 8t + 2t^2) - 3} - (6x + 5t + 9)$ for a real variable x.

It holds:

- $|E(H_{t,a})| - u_0(|V(H_{t,a})|) \geq f_t(a)$ for every positive integer a,
- $f_t(x)$ is increasing for every $x > 0$ and $t > 0$.

Therefore $|E(H_{t,a})| - u_0(|V(H_{t,a})|) \geq t$.

□

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Proposition

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Corollary

For every integer t, there are infinitely many values n such that $u_1(n) - u_0(n) \geq t$. Hence $\limsup_{n\to\infty}(u_1(n) - u_0(n)) = \infty$.

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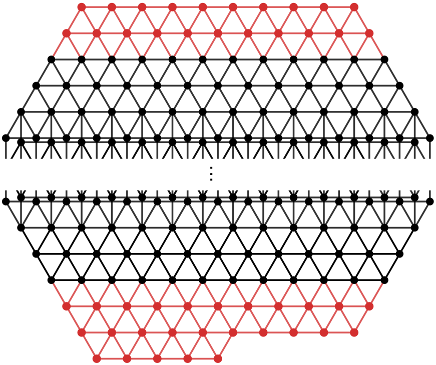
Proof:

Fix an even integer t, $t \geq 8$ and examine the graphs $H_{t,a}$ and $H_{t,a+1}$.
For $n \in \{|V(H_{t,a})|, |V(H_{t,a+1})|\}$ construct a graph $L_{t,n}$:



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The number of vertices that separate $H_{t,a}$ from $H_{t,a+1}$ is

$$|V(H_{t,a+1})| - |V(H_{t,a})| = 3a^2 + 15a + 19 + 6at + 14t + 2t^2 -$$
$$-(3a^2 + 9a + 6at + 7 + 8t + 2t^2) = 6a + 6t + 12.$$

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$$- (3a^2 + 9a + 6at + 7 + 8t + 2t^2) = 6a + 6t + 12.$$

Suppose further that $a \geq t^2$. Attaching paths of triangles T_{a+1+t}, T_{a+t} and T_{a+t-1} to the upper horizontal side of $G_{t,a}$, and T_{a+1+t}, T_{a+t}, T_{a+t-1} and T_{a+t-2} to the bottom horizontal side of $H_{t,a}$. This operation adds:

$$2(a + 1 + t + (a + t) + (a + t - 1)) + (a + t - 2) = 7a - 2 + 7t =$$
$$= 6a + a - 2 + 6t + t \geq 6a - 2 + 6t + 3t > 6a + 6t + 13$$

vertices, which is more than enough.

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A careful computation shows that

$$|E(L_{t,n})| - u_0(n) \geq |E(H_{t,a})| - u_0(|V(H_{t,a})|) - 7 \geq t - 7 \geq 1.$$

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Substituting the smallest values for a and t, which means $t = 8$ and $a = 64$, we get

$$|V(H_{8,64})| = 16135.$$

$\square$

Proof of Theorem 2

Theorem 2

For any constant $\alpha < \sqrt[4]{\frac{1}{3}} = .7598\ldots$ and for every n sufficiently large (with respect to α), $u_1(n) - u_0(n) \geq \alpha\sqrt[4]{n}$.

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Proof:

Denote $n(t,a) := |V(H_{t,a})|$ and extend the function $n(t,t^2) = 3t^4 + 6t^3 + 11t^2 + 8t + 7$ to all positive real values of t. Then t can be expressed as

$$t = \sqrt{\sqrt{\frac{n}{3} - \frac{5}{9}} - \frac{39}{36}} - \frac{1}{2}.$$

It follows that $t \geq \beta\sqrt[4]{n}$ holds for sufficiently large n (and hence also for sufficiently large t), as long as $\beta < \sqrt[4]{\frac{1}{3}}$.

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Let $t_0 \geq 8$ be an even integer large enough, so that

$$t_0 - 7 \geq \frac{\alpha}{\beta}(t_0 + 2) \text{ and } t_0 \geq \beta \sqrt[4]{n(t_0, t_0^2)}.$$

(Since the involved functions are monotone, it follows that

$$t - 7 \geq \frac{\alpha}{\beta}(t + 2) \text{ and } t \geq \beta \sqrt[4]{n(t, t^2)}$$

hold for every $t \geq t_0$.)

Proof of Theorem 2

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For any constant $\alpha < \sqrt[4]{\frac{1}{3}} = .7598\ldots$ and for every n sufficiently large (with respect to α), $u_1(n) - u_0(n) \geq \alpha \sqrt[4]{n}$.

For every $n \geq n(t_0, t_0^2)$, there is an even integer t such that $n(t, t^2) \leq n < n(t+2, (t+2)^2)$. Then, using this t, it follows from Theorem 1, that

$$u_1(n) - u_0(n) \geq t - 7 \geq \frac{\alpha}{\beta}(t+2) \geq \frac{\alpha}{\beta}\beta \sqrt[4]{n(t+2, (t+2)^2)} \geq \alpha \sqrt[4]{n}.$$

$\square$

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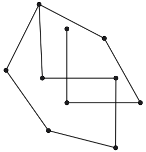
Define $u_{0.5}(n)$ as the maximum number of edges of an n-vertex graph that allows for 1-planar unit distance drawings in which only vertical and horizontal edges cross.



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Prove an upper bound on $u_{0.5}(n)$ better than $3n - c\sqrt[4]{n}$.

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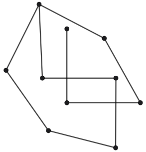

Problem 2.

Prove an upper bound on $u_{0.5}(n)$ better than $3n - c\sqrt[4]{n}$.

Problem 3.

Is $u_{0.5}(n) < u_1(n)$ for some n? For infinitely many n's?

Thank you for your attention!