Connectivity Augmentation for Planar and Beyond-Planar Graphs

Michael Kaufmann
with Hugo Akitaya, Justin Dallant, Erik Demaine, Linda Kleist, Frederick Stock, Csaba Tóth, Torsten Ueckerdt

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Previous work

Therese Biedl, Goos Kant, MK: On triangulating planar graphs under the four-connectivity constraints, Algorithmica 19(4), 427-446, 1997.

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Still very often:

assume triangulated planar graph
assume 3-connectivity
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Nowadays ..

go beyond-planar $\rightarrow$ 1-planar, 2-planar, etc.
consider higher connectivity than 4

Models

Distinguish

- geometric model (straight-line drawing)
- topological model (embedding)
- abstract model

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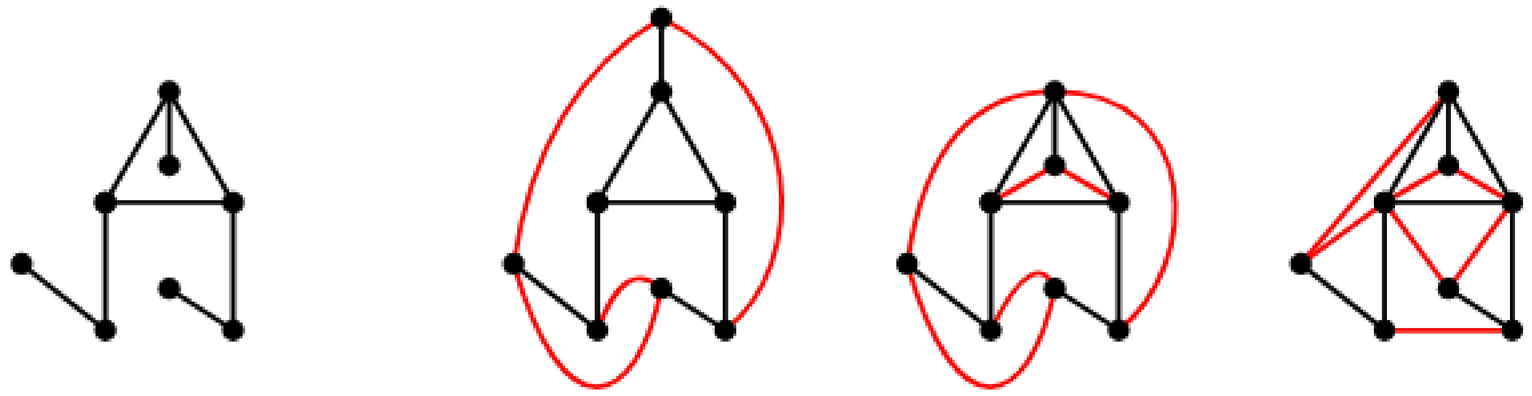

Minimum augmentation to 3-connectivity (4,6,7 edges)

General remarks

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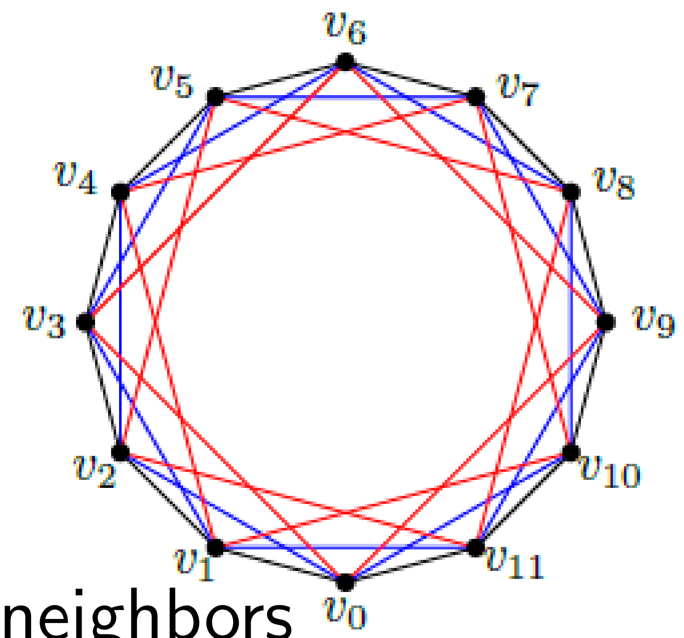
Open: Precise dependency, say for 2-planarity

Geometric setting

n **points in convex position:**



k**-circulant graph:**
vertices connected to their k next neighbors

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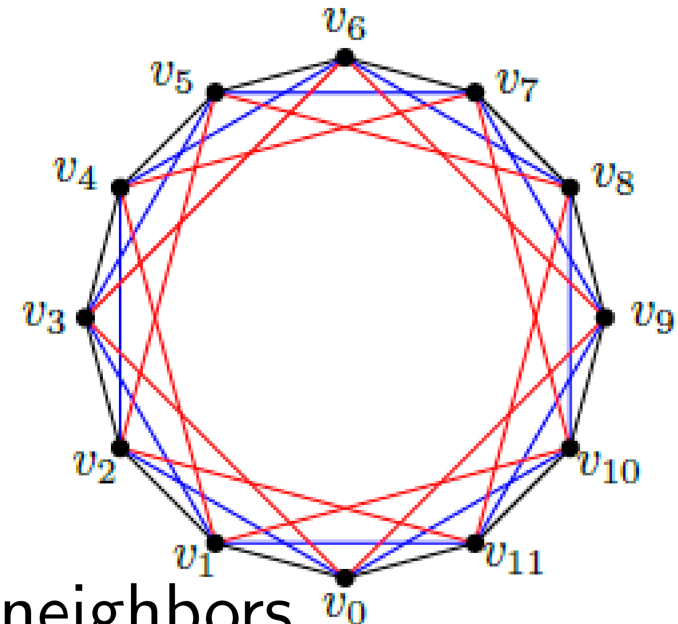

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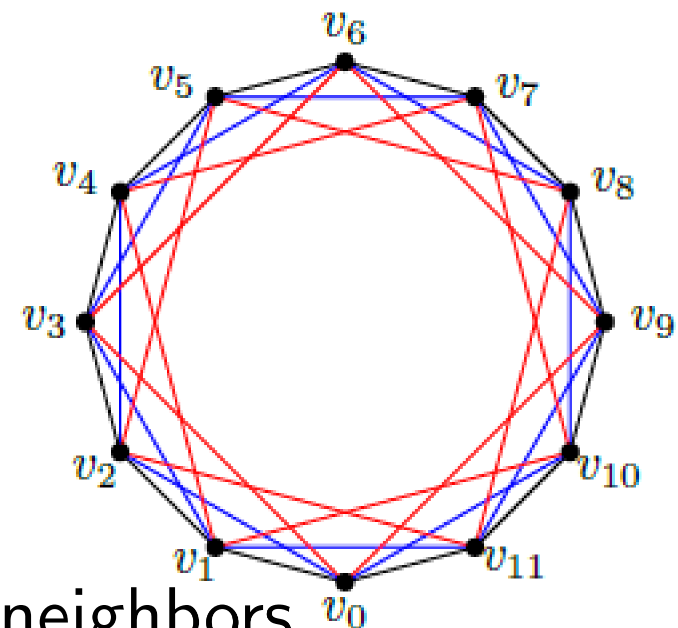

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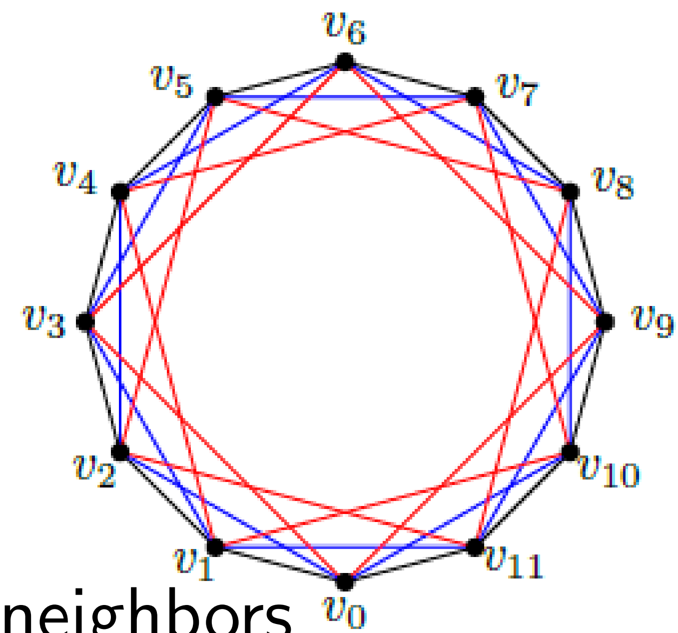

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Proof idea: Let $v_0, \ldots, v_n$ be vertices in ccw order
Consider edge (v_0, v_ℓ).
Estimate how many edges start at $v_1, \ldots, v_\ell$ and end outside!
This gives lower bound for lcr

topological graphs

Task: Augment 3-connected planar graph to 4-connectivity

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Apply a known result about flipping triangles:

Bose et al. Triangulation can be converted into a 4-connected triangulation by a single simult. flip of $\leq \frac{2}{3}n$ edges.

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both problems NP-complete

Some complexity results

Theorem: The min. augmentation problem for a c-connected planar graph G to a k-connected planar graph G' is NP-hard, for $2 \rightarrow 3, 3 \rightarrow 4, 4 \rightarrow 5$.

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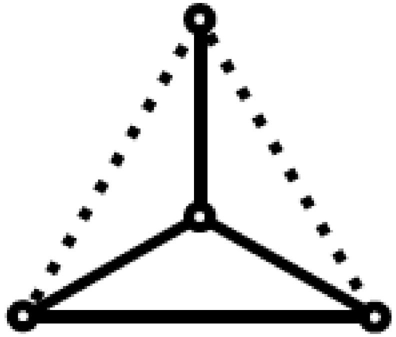
Problem stays hard if G' might be 1-planar (topol./geometric)

Some ideas for $3 \to 4$

The building block



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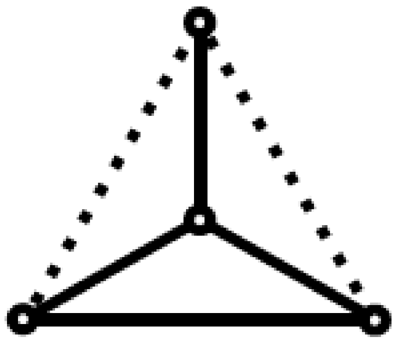
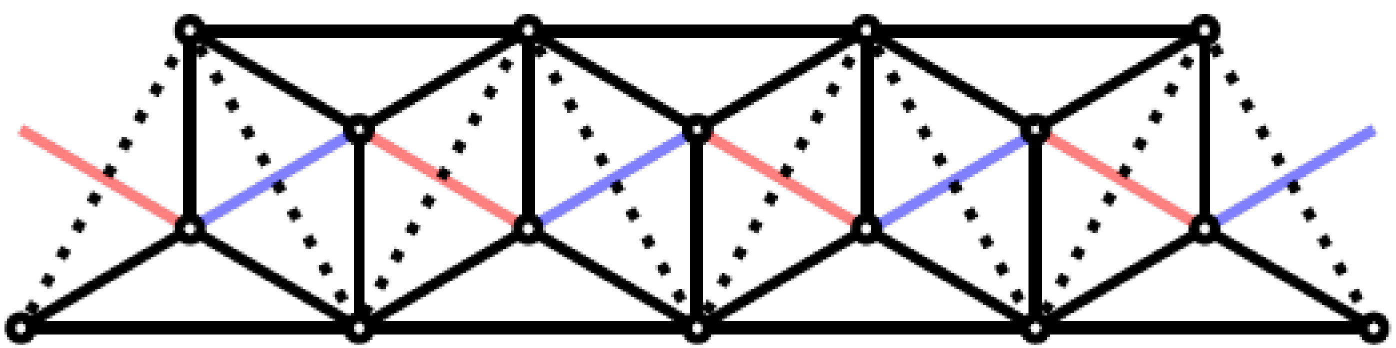


The variable gadget



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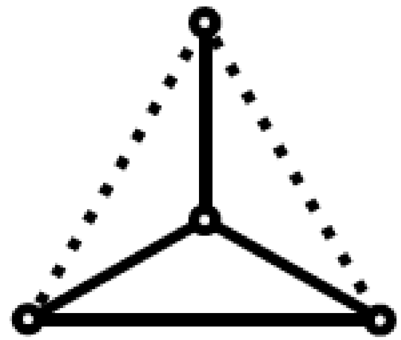

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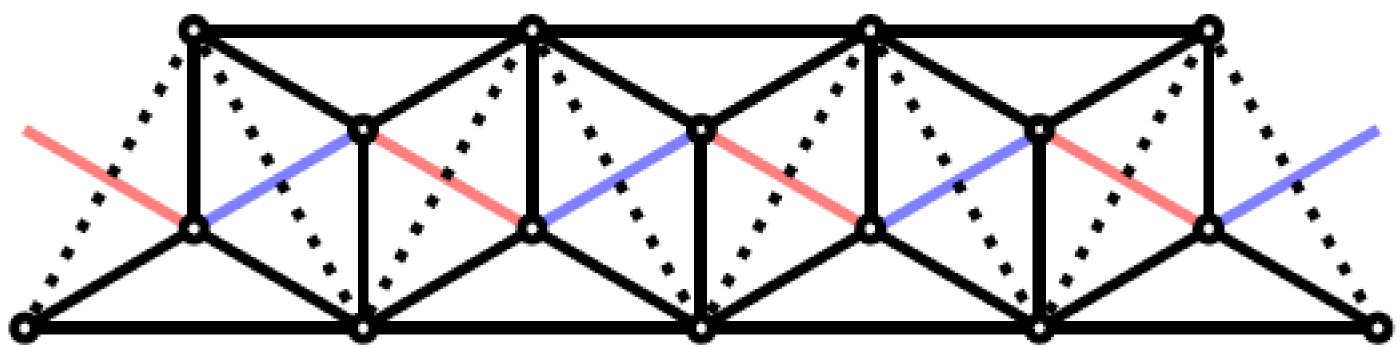
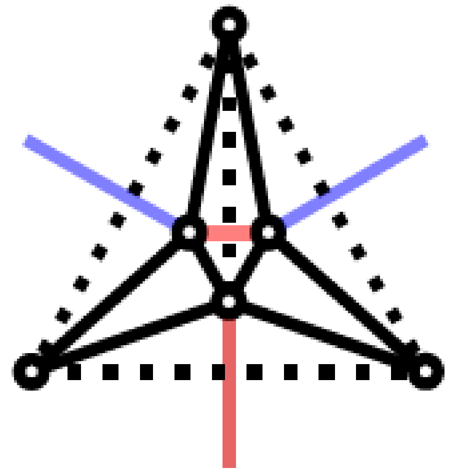


min. augmentation by either red or blue edges !

$3 \to 4(2)$

A positive literal gadget



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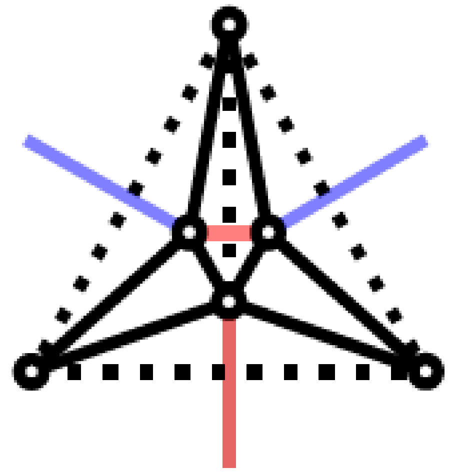

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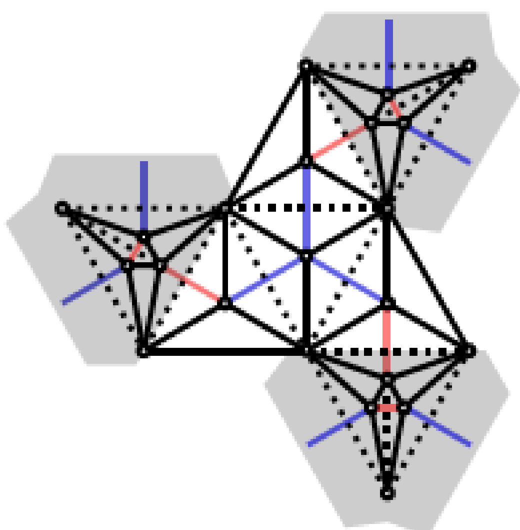

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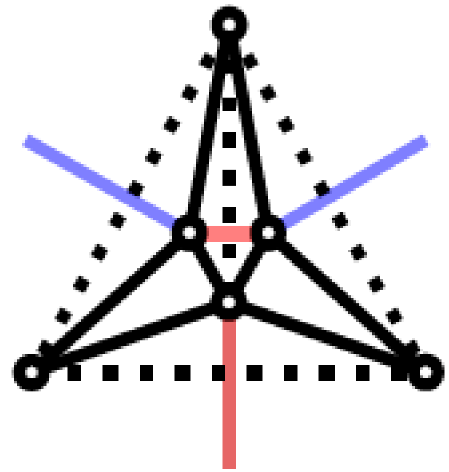

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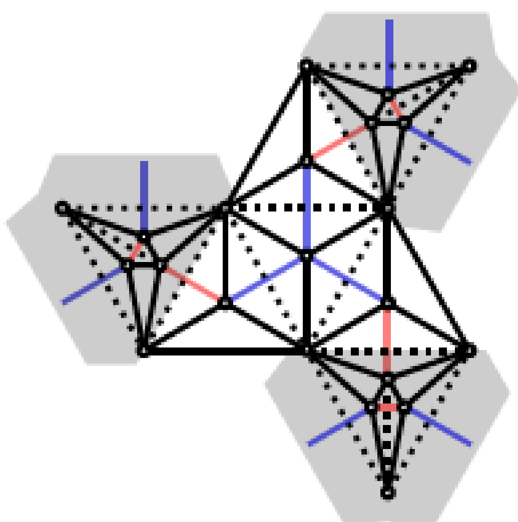
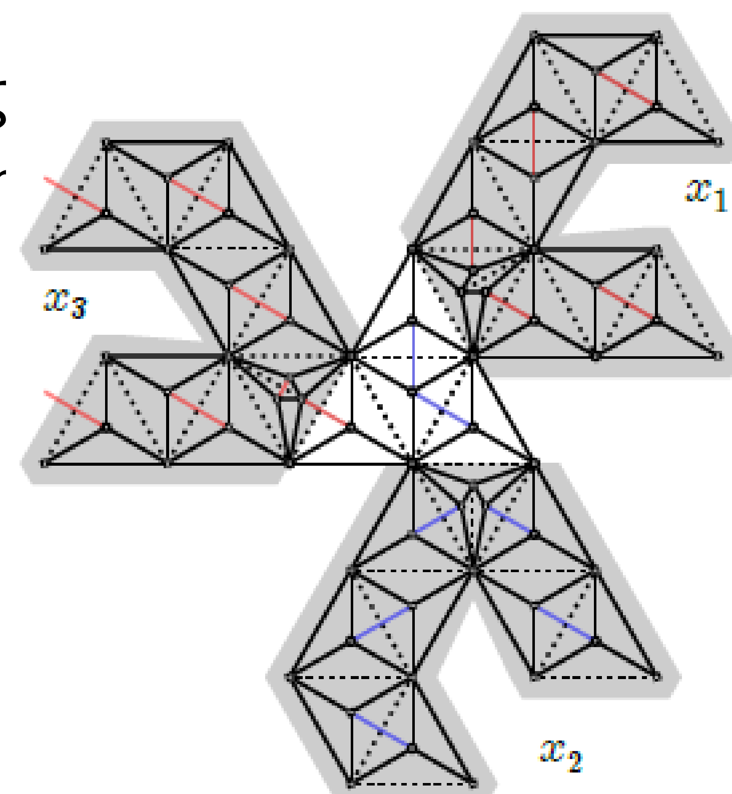


Satisfying
assignmer



Doable cases

Theorem: Given a plane tree T, we can compute a minimum augmentation to a plane 3-connected graph G, where T is spanning for G.

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Theorem: Any planar straight-line graph of n points in convex position can be augmented to a 3-connected 5-planar graph. '5' is best possible.

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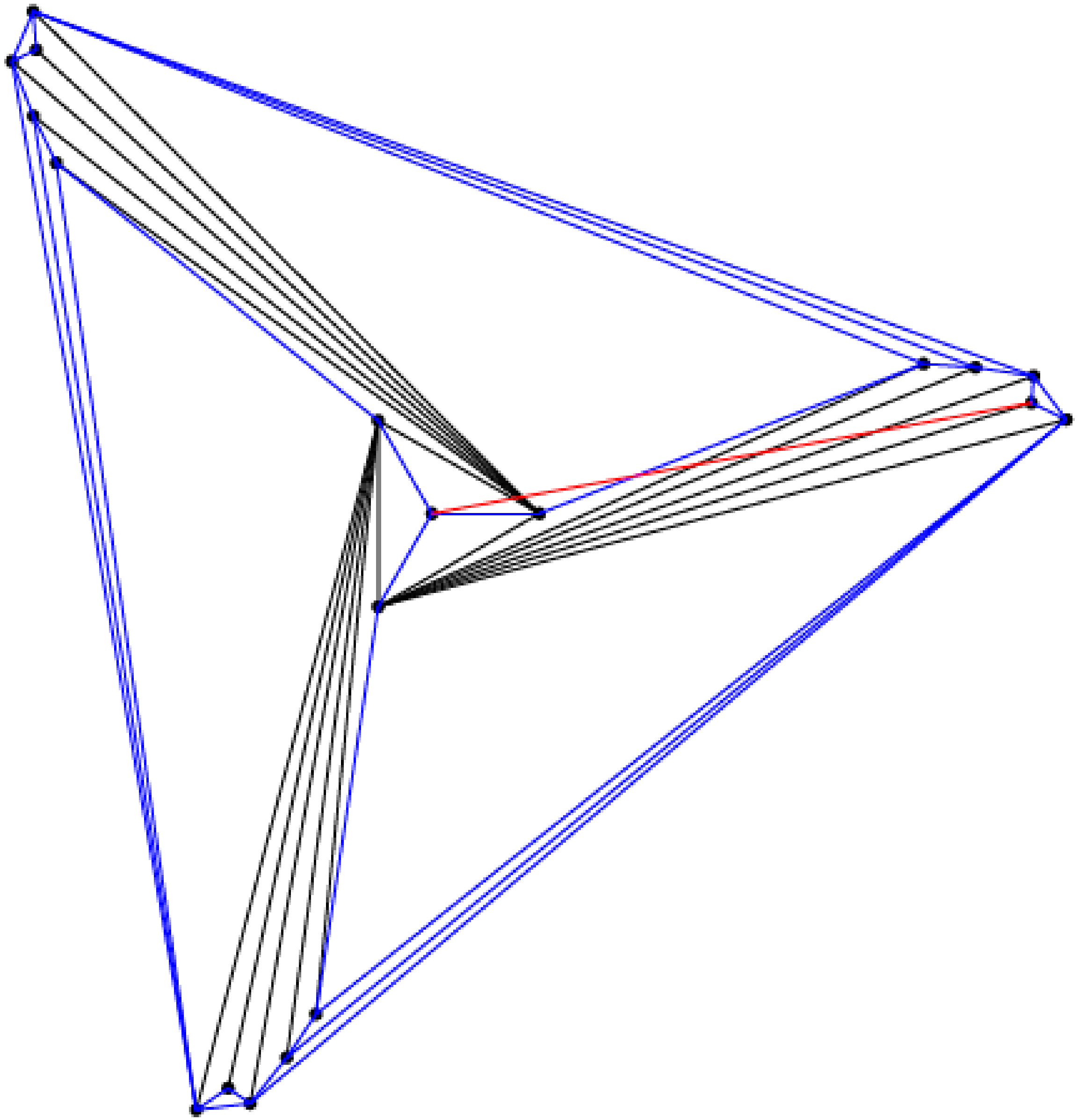
Theorem: Any planar straight-line graph of n points in convex position can be augmented to a 3-connected 5-planar graph. '5' is best possible.

Theorem: Any PSLG of n points in convex position can be augmented to a k-connected $O(k^2)$-planar graph.
This is best possible.

One last remarkable result



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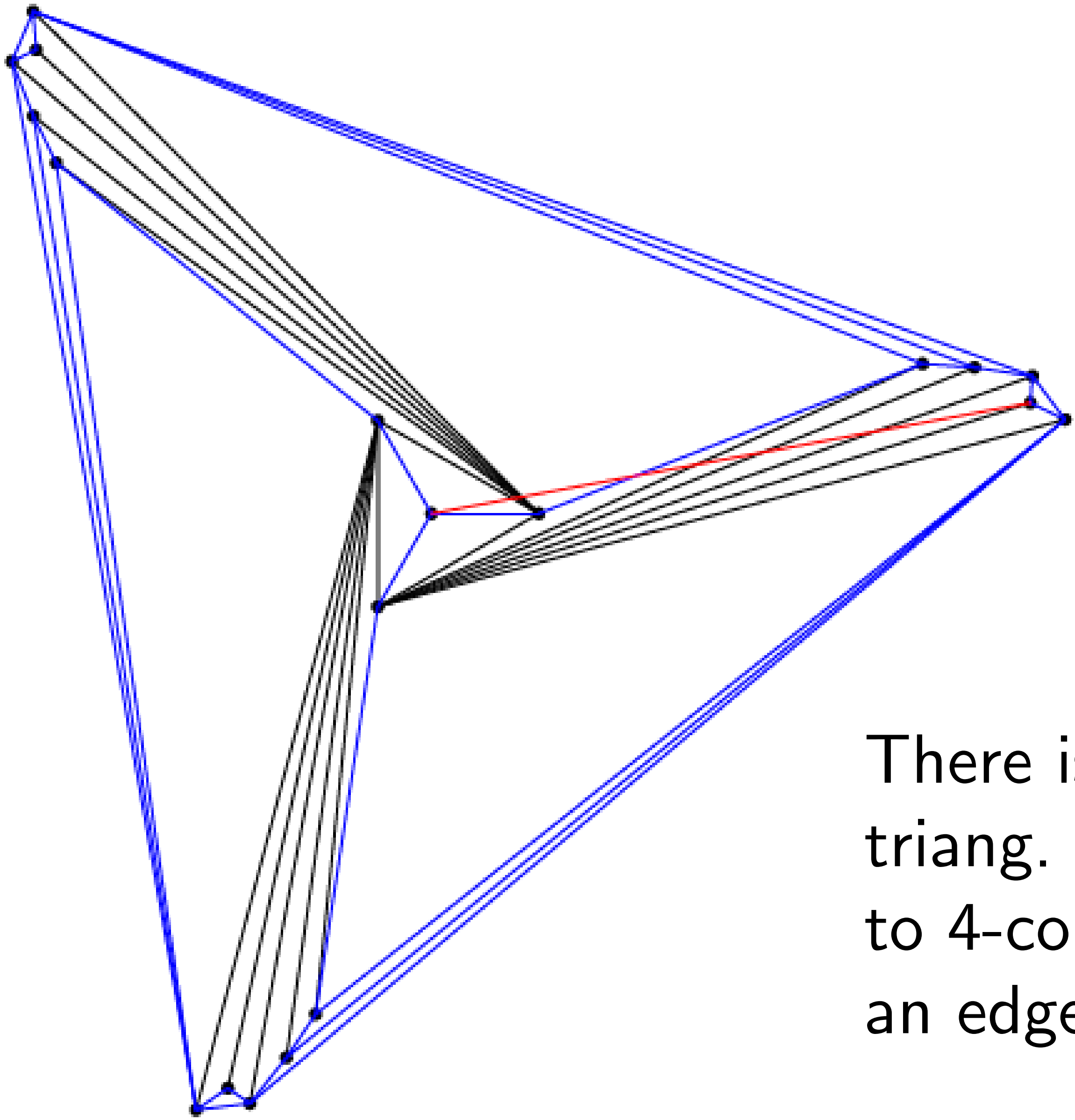

There is a geometric triang. s.t. any augmentation to 4-connect. has an edge with $\Omega(n)$ crossings