slides available

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Structural Parameterizations of k-Planarity

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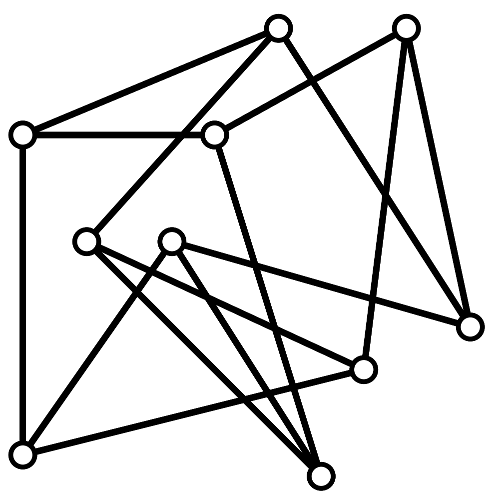
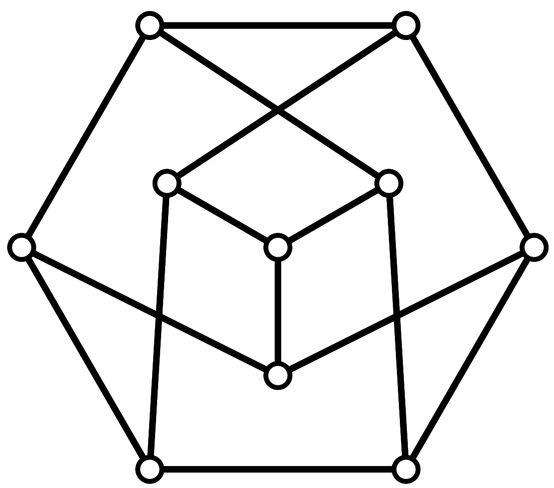
Definition: k-Planarity

A *k*-planar drawing is a drawing of a graph in the plane such that every edge involves at most k crossings.

A graph is *k*-planar if it admits a k-planar drawing.



3-planar



1-planar

The local crossing number of a graph G,
$\mathrm{lcr}(G)$ is the minimum k such that G is a k-planar graph.

Testing k-Planarity is Hard

We know that computing lcr(G) is very hard.

- Testing 1-planarity is NP-complete,
  [Grigoriev & Bodlaender, Algorithmica 2007]

- even on near-planar graphs with max degree 20.
  (a planar graph + a single edge)
  [Cabello and Mohar, SoCG 2010; SIAM J. Comput. 2013]

- Testing k-planarity is NP-complete for every $k \geq 1$, even if lcr(G) $\leq k$ or lcr(G) $\geq 2k$ is guaranteed.
  - no poly-time $(2 - \varepsilon)$-approximation, unless P=NP.

  [Urschel and Wellens, IPL 2021]

Parameterized Complexity of 1-Planarity

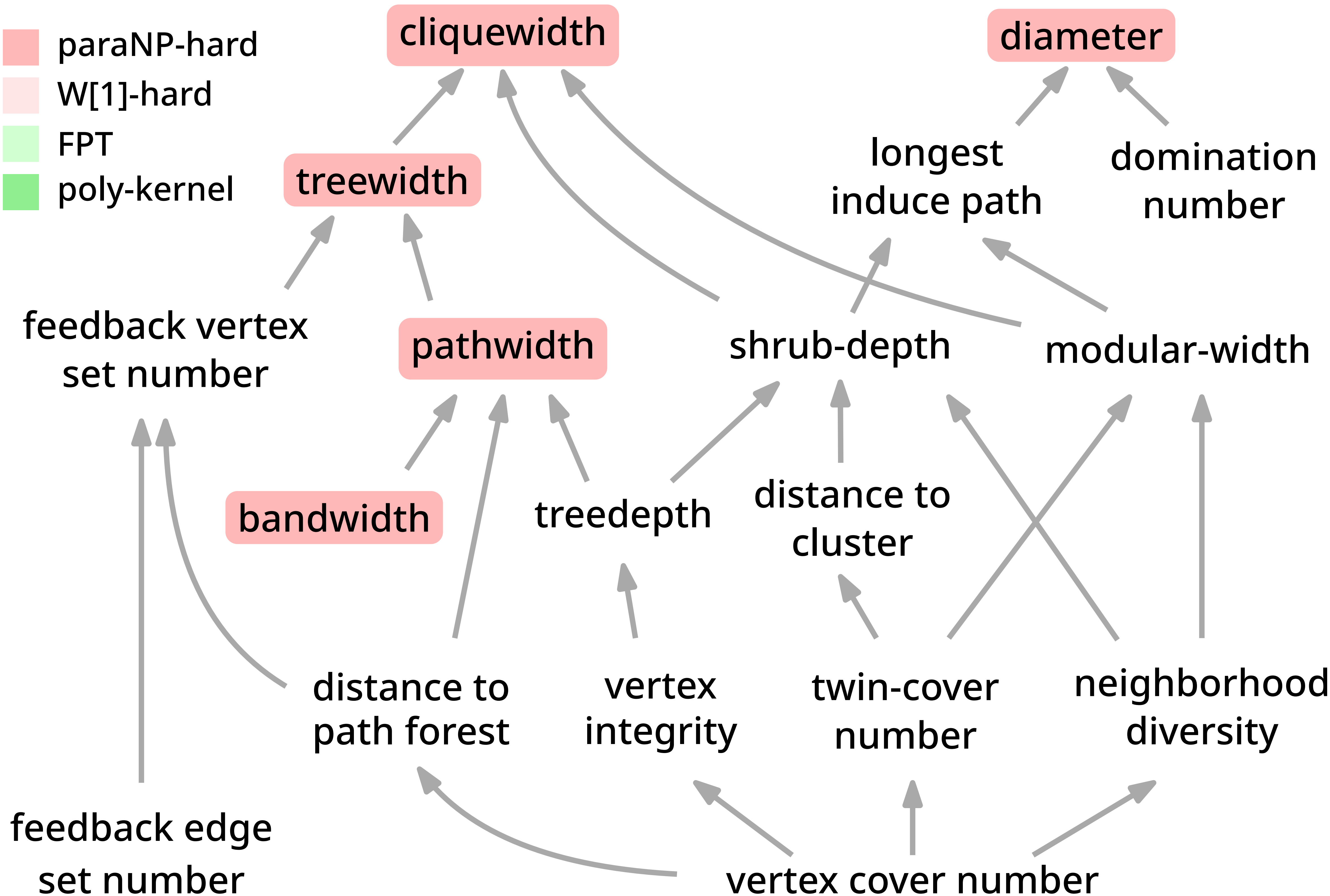
To overcome the hardness,
Bannister, Cabello, and Eppstein [WADS 2013; JGAA 2018]
initiated structural parameterizations of 1-planarity.

They showed that testing 1-planarity is:

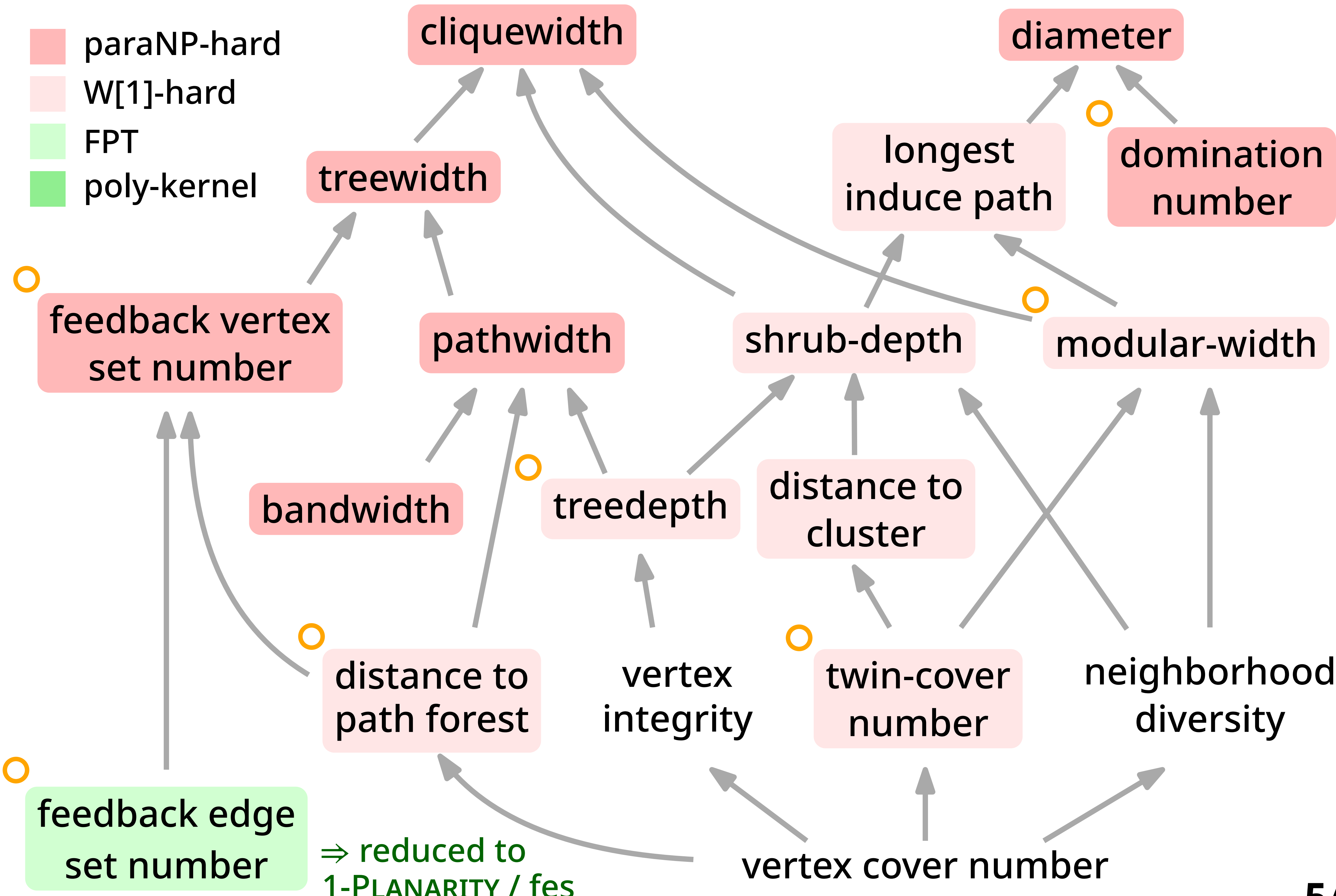
- FPT w.r.t.
  - feedback edge set number
  - treedepth
  - vertex cover number (poly-kernel)
- paraNP-complete w.r.t.
  - bandwidth

We extend the results to *k*-planarity and other params.

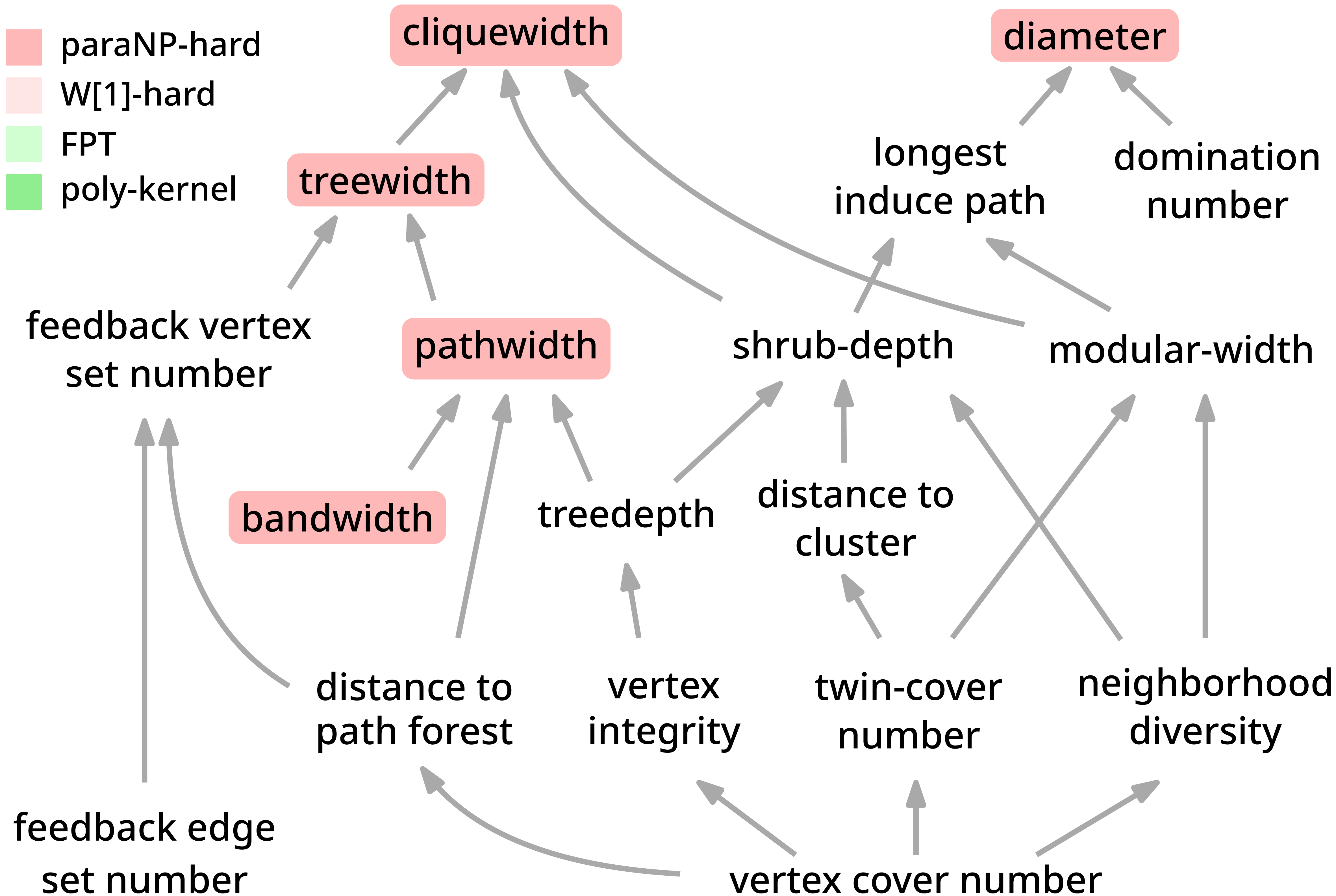
Known Results



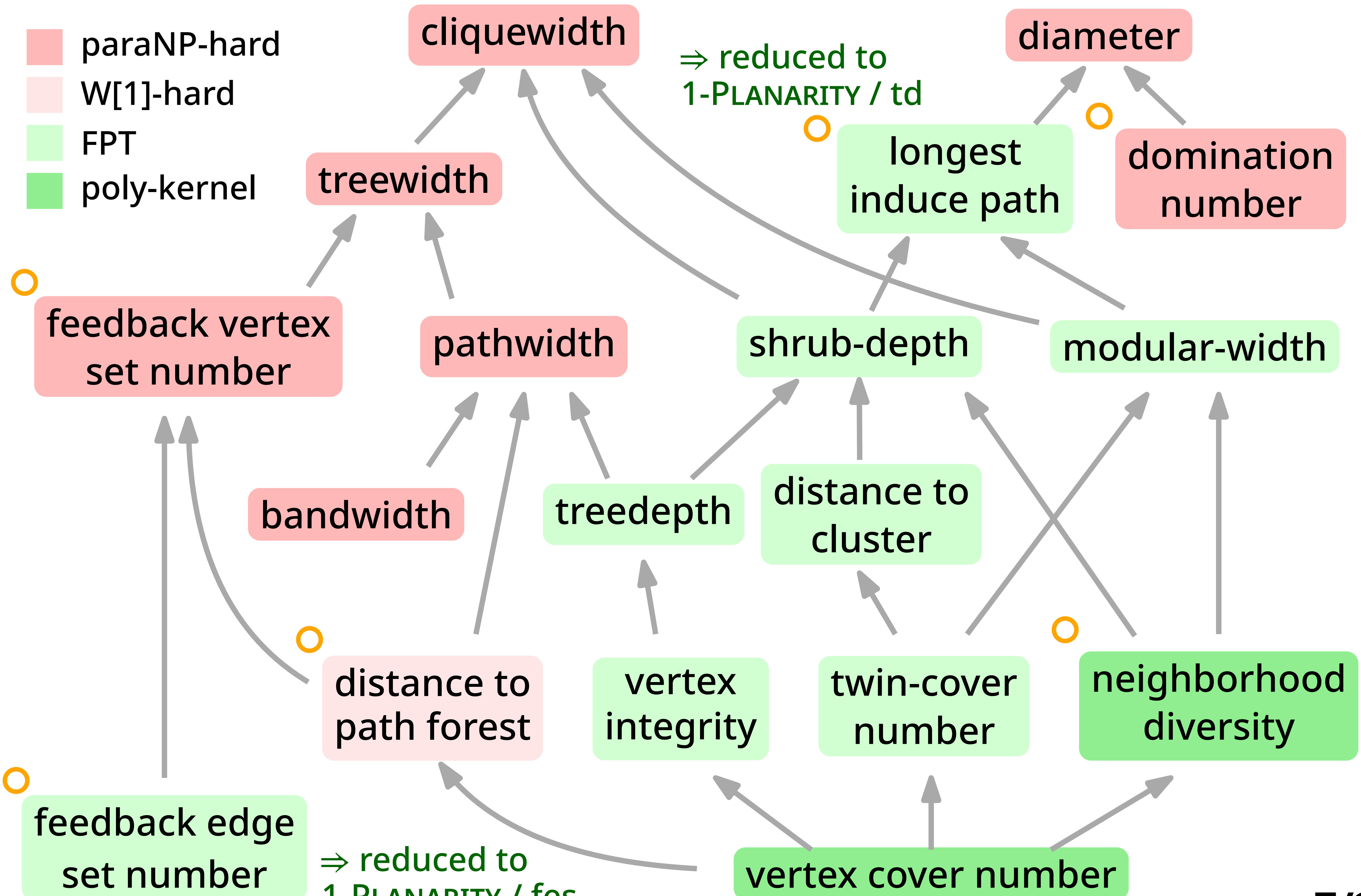
Our Results


paraNP-hard
W[1]-hard
FPT
poly-kernel
cliquewidth
diameter
treewidth
longest induce path
domination number
feedback vertex set number
pathwidth
shrub-depth
modular-width
bandwidth
treedepth
distance to cluster
distance to path forest
vertex integrity
twin-cover number
neighborhood diversity
feedback edge set number
⇒ reduced to 1-PLANARITY / fes
vertex cover number

Known Results (+k Setting)



Our Results (+k Setting)



Our Selected Results

To the GD community, I want to advertise:

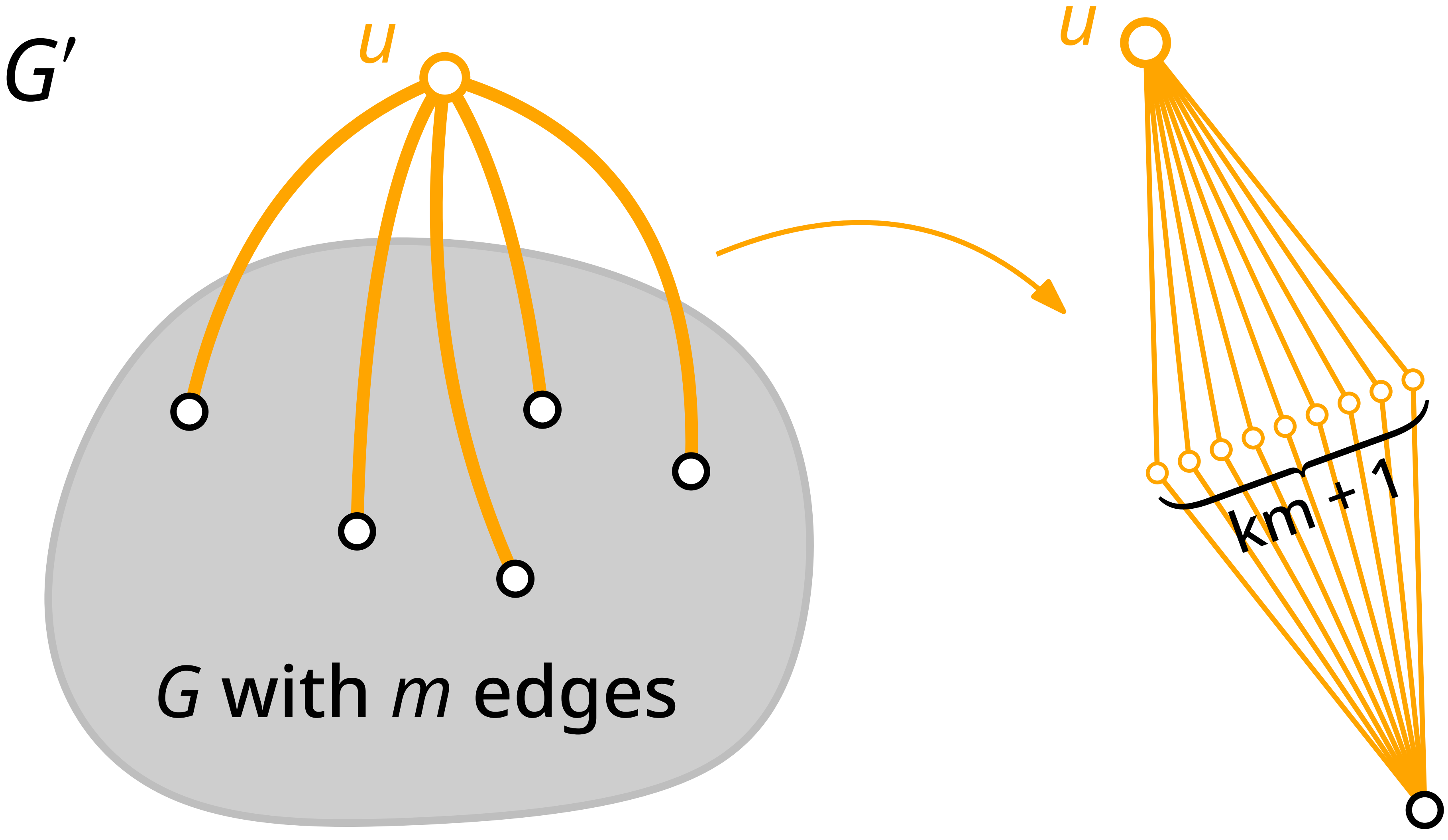
- Testing 1-planarity is NP-complete,
  even on near-planar graphs with
  - pathwidth at most 4, and
  - feedback vertex set number (fvs) at most 3.

- For every $c \geq 1$, testing k-planarity is NP-complete, even if $\mathrm{lcr}(G) \leq k$ or $\mathrm{lcr}(G) \geq ck$ is guaranteed.
  - no constant-factor approximation, unless P=NP.

Thick Edges

Consider G' consisting of

- a graph G with m edges
- a vertex u adjacent to some vertices via **thick edges**.



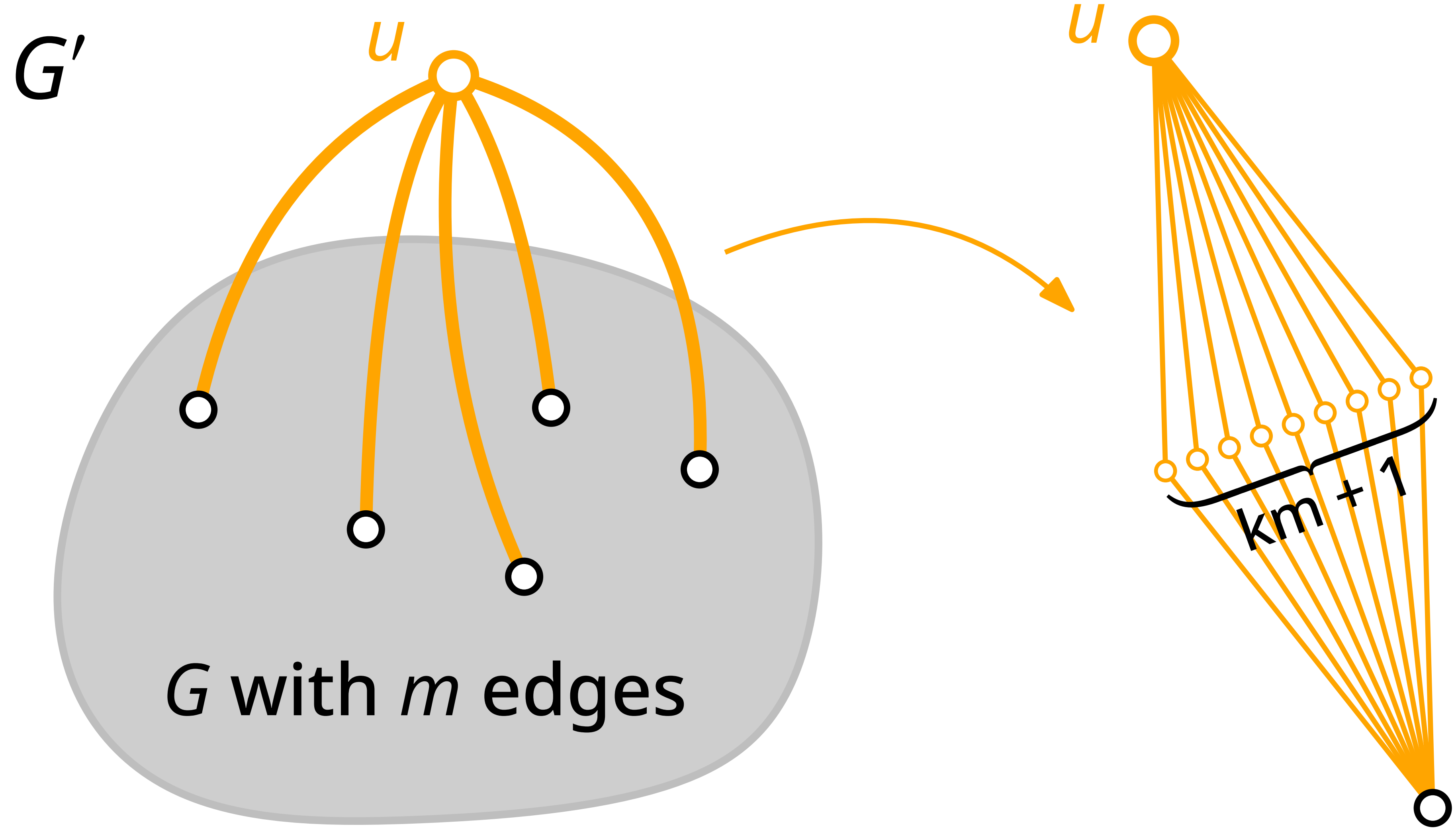
Each **thick edge** consists of $km + 1$ paths of length 2.

Property of Thick Edges

Since G can have at most km crossings,

Observation

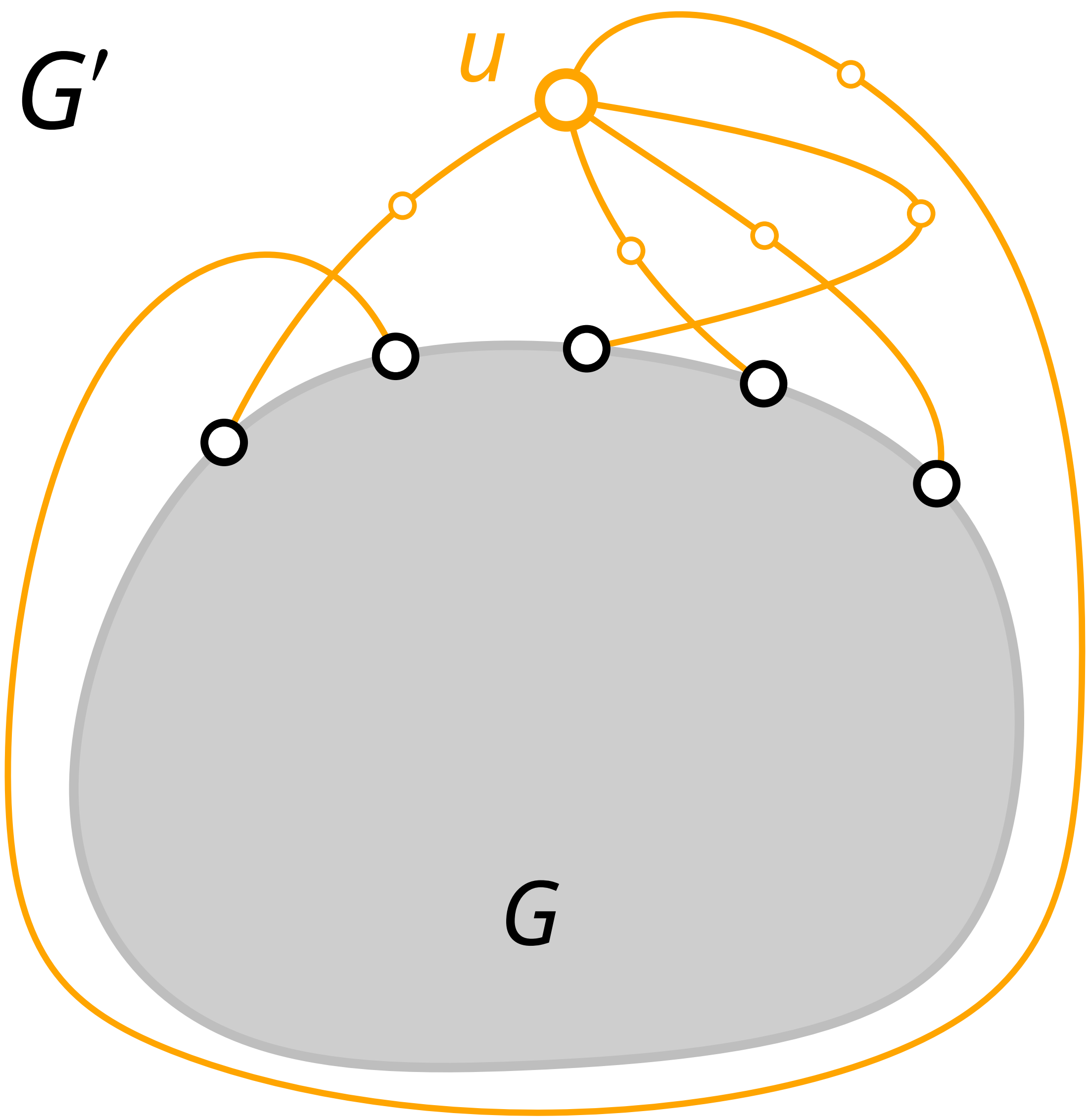
In any k-planar drawing of G', every **thick edge** contains a path that is not crossed by an edge of G.



Property of Thick Edges

Lemma

If G' is k-planar, there exists a k-planar drawing of G' such that **the thick edges** do not involve a crossing.



- for each **thick edge**, take a path not crossing G.

Property of Thick Edges

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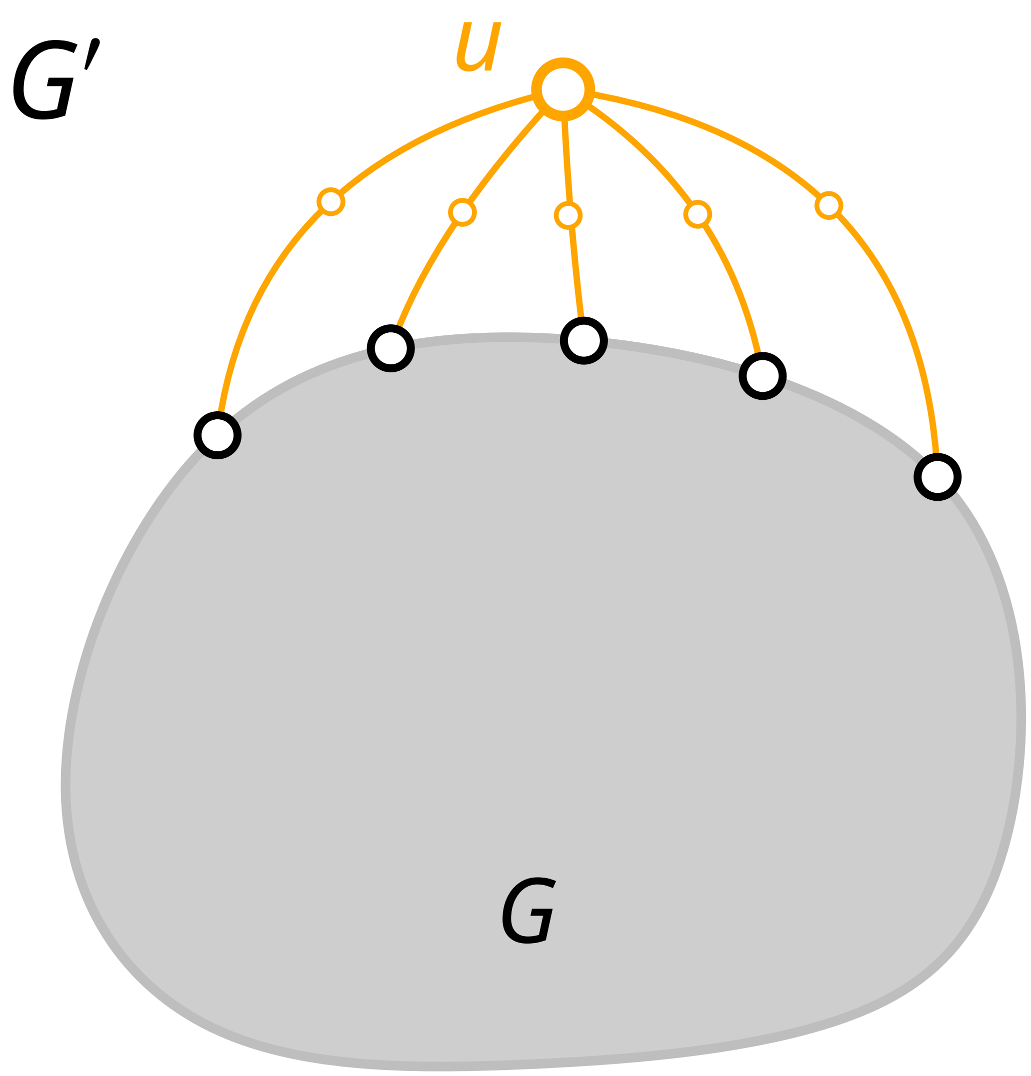

- for each **thick edge**, take a path not crossing G.
- untangle them.

Property of Thick Edges

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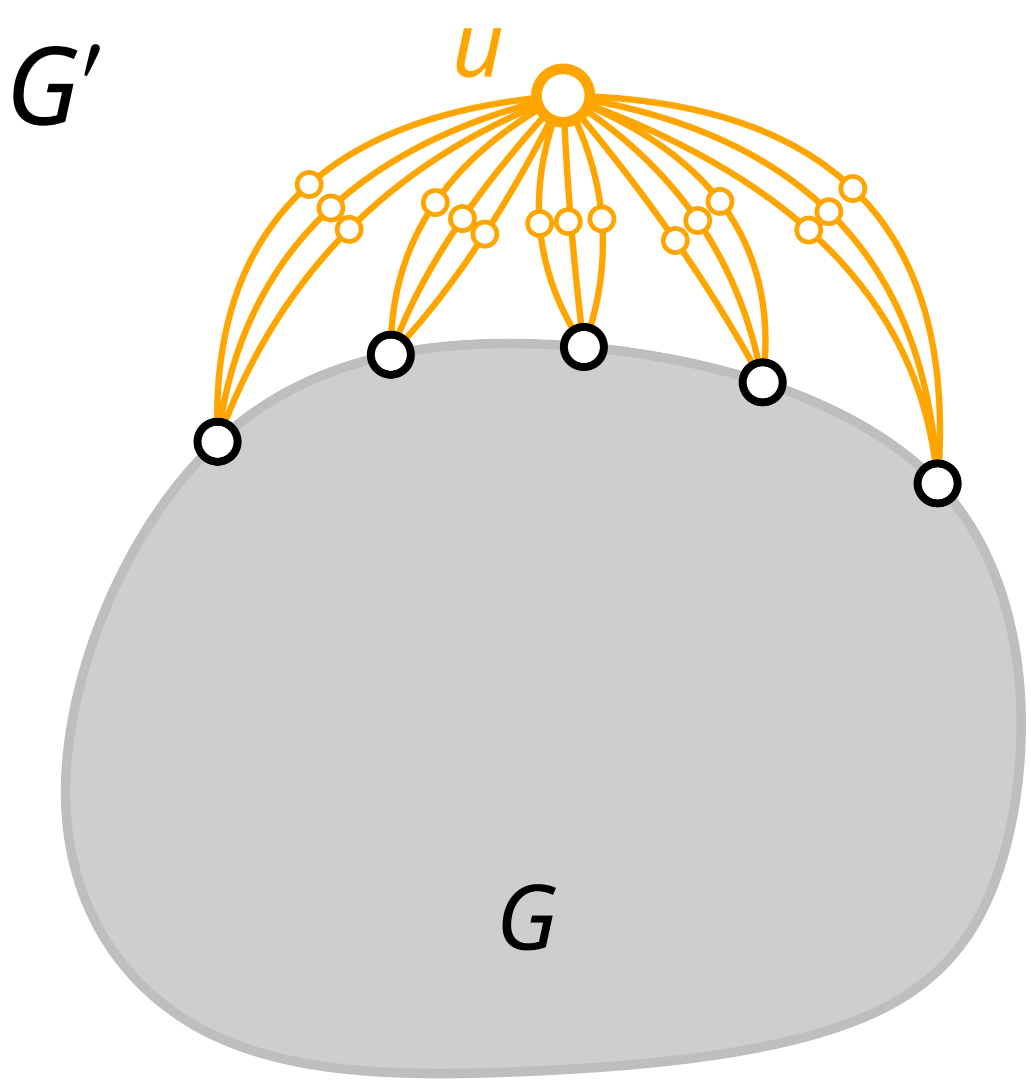


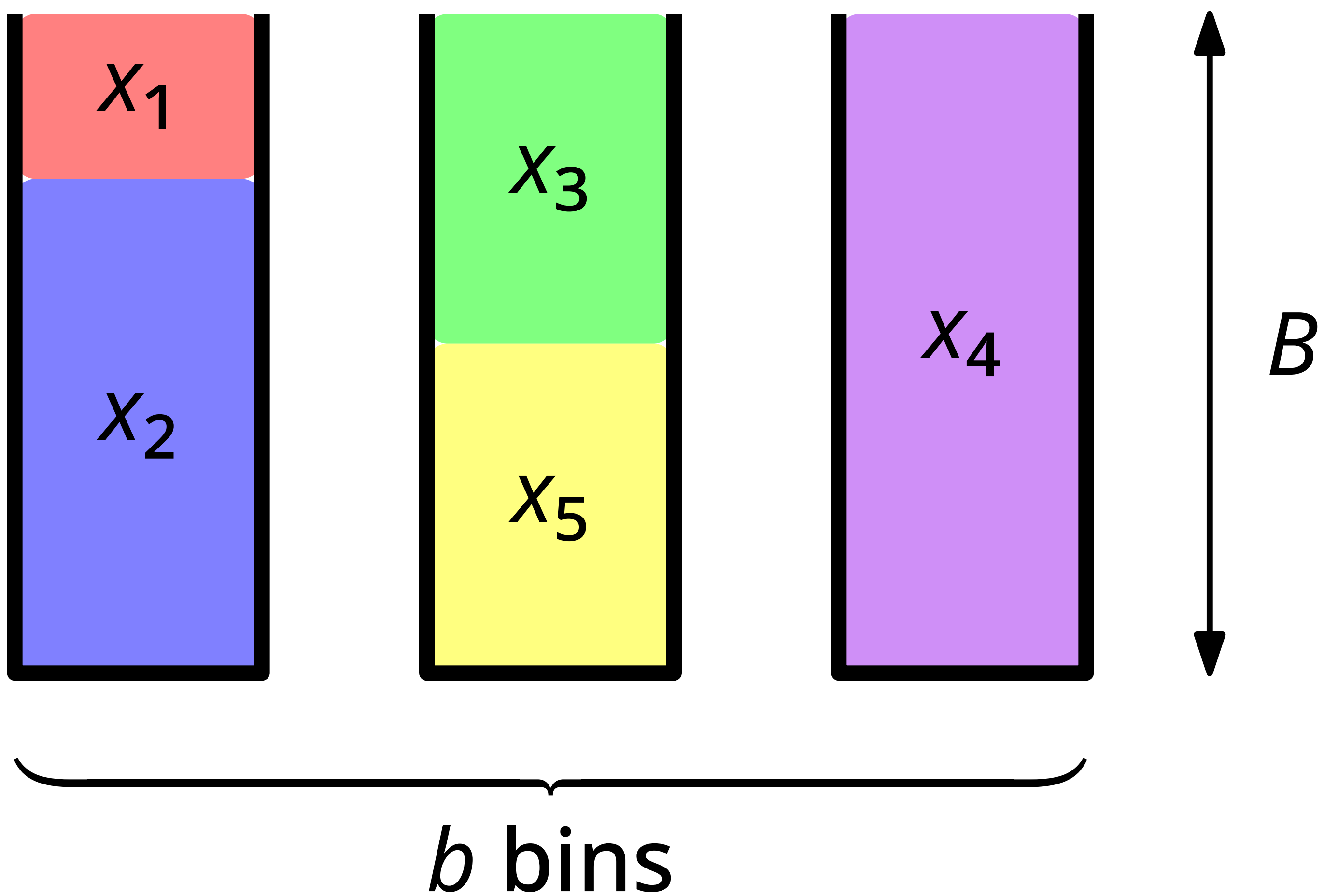
- for each **thick edge**, take a path not crossing G.
- untangle them.
- redraw the other paths.

Reduction Type 1: from Unary Bin Packing

Problem: UNARY BIN PACKING

Input: integers $x_1, \dots, x_n$ (≥ 0), b, B. (unary encoded)
Question: Can we partition $\{x_1, \dots, x_n\}$ into b sets, so that the sum of each set is B?

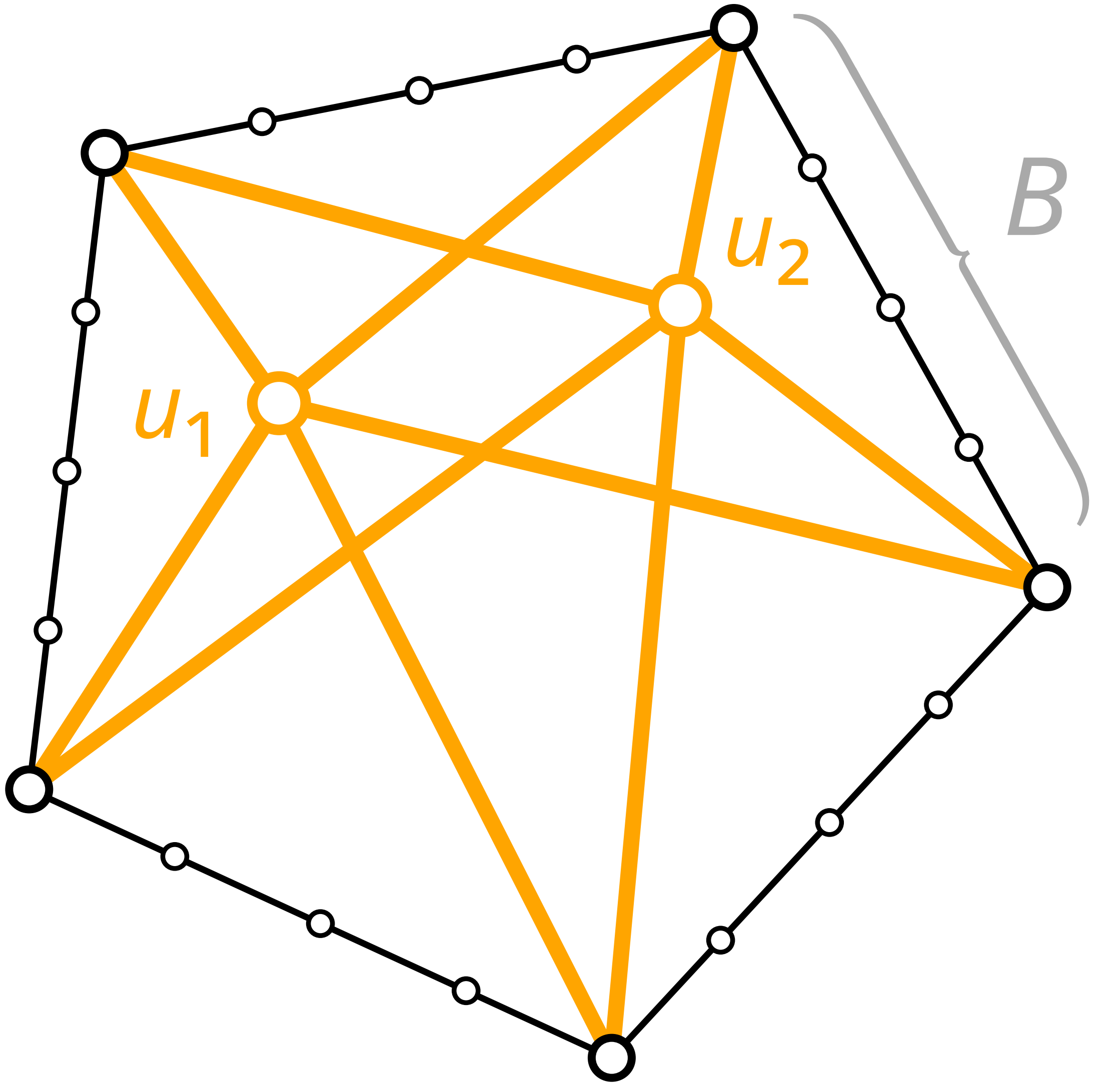
W[1]-hard / b.
[JKMS, JCSS 2013]



Type 1: Construction

We reduce Unary Bin Packing to 1-Planarity.

bins

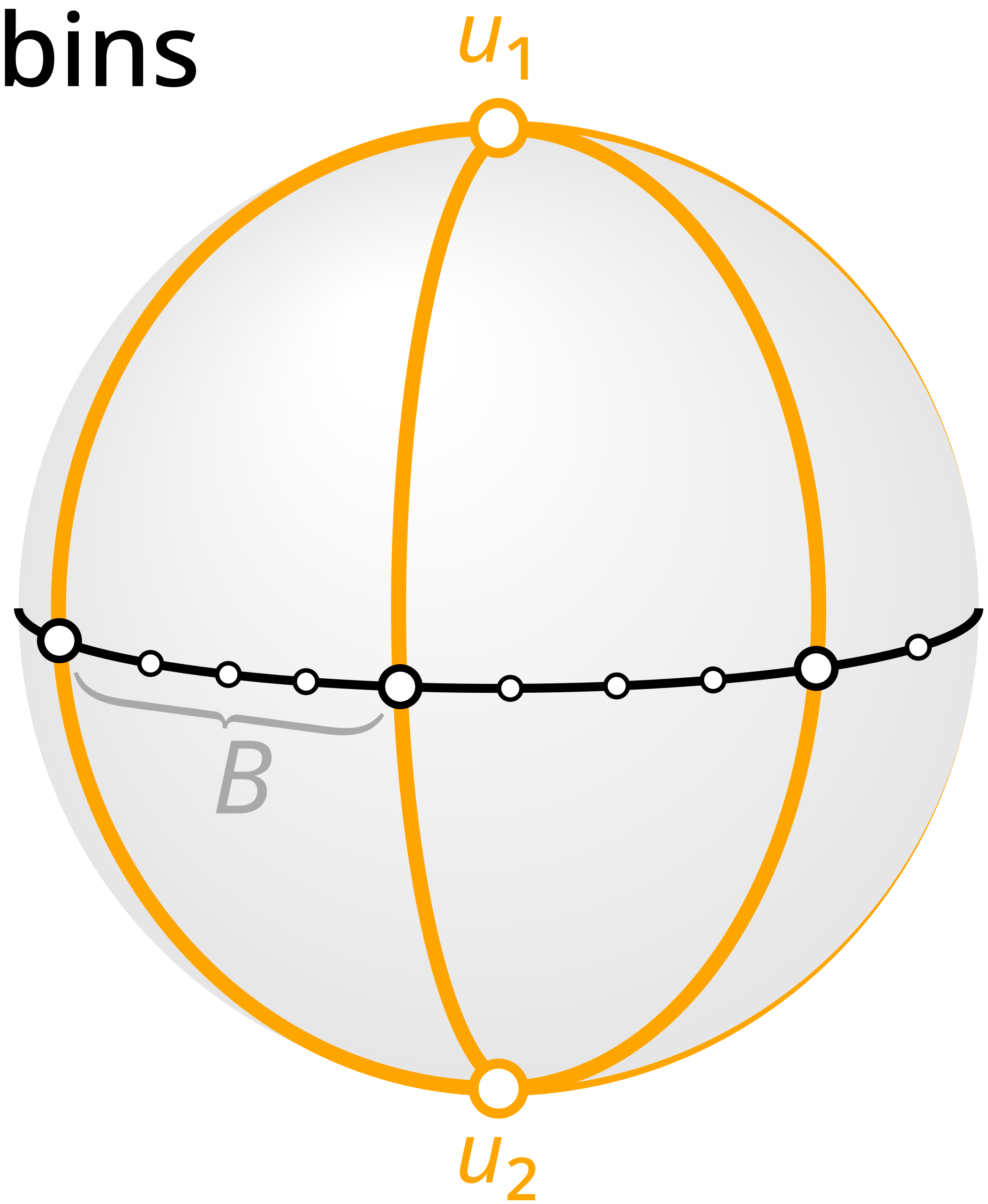


- b-gon where every edge is a path of length B.
- u_1, u_2 with thick edges connected to b corners.

Type 1: Construction

We reduce UNARY BIN PACKING to 1-PLANARITY.

bins



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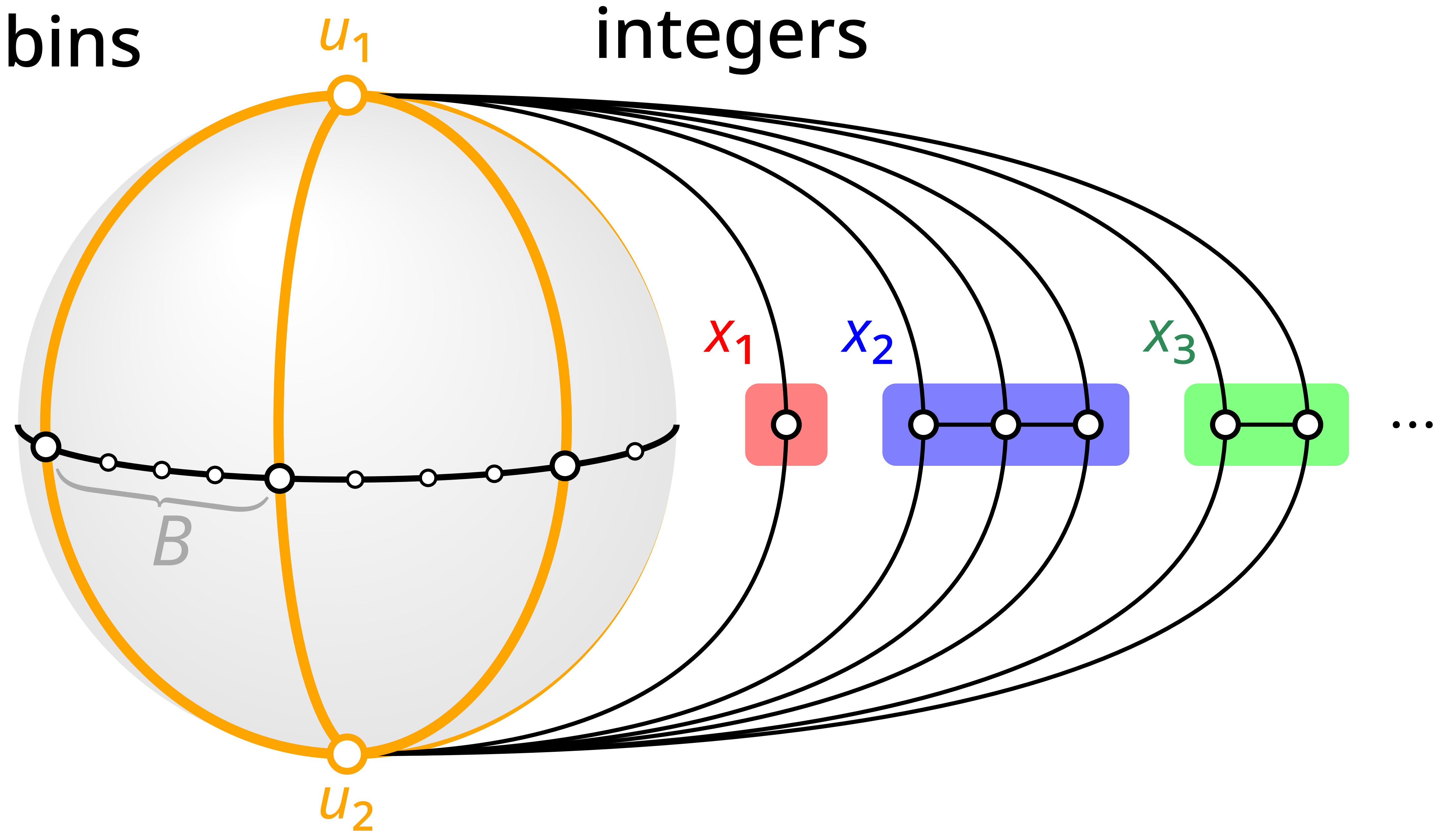

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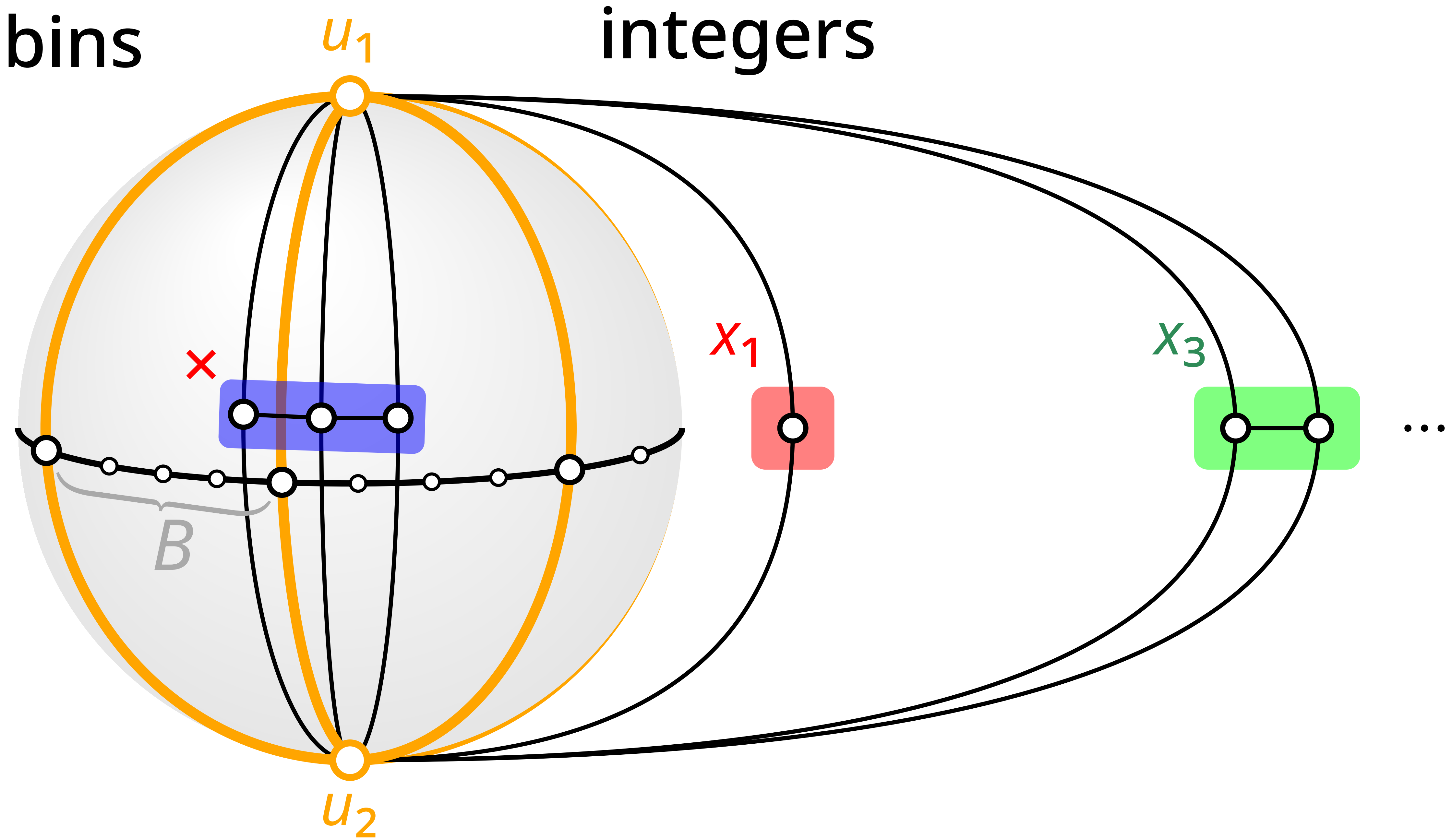

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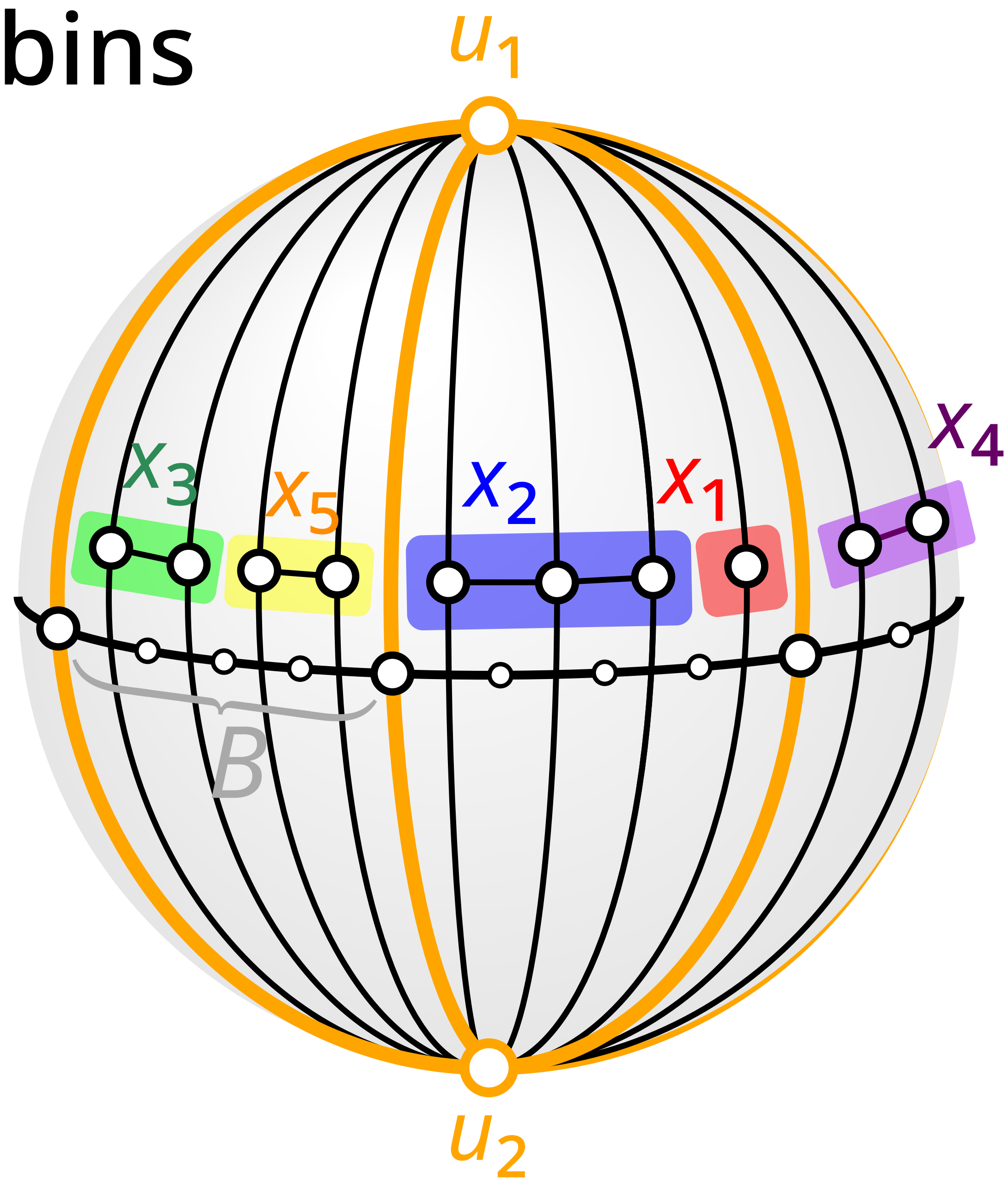
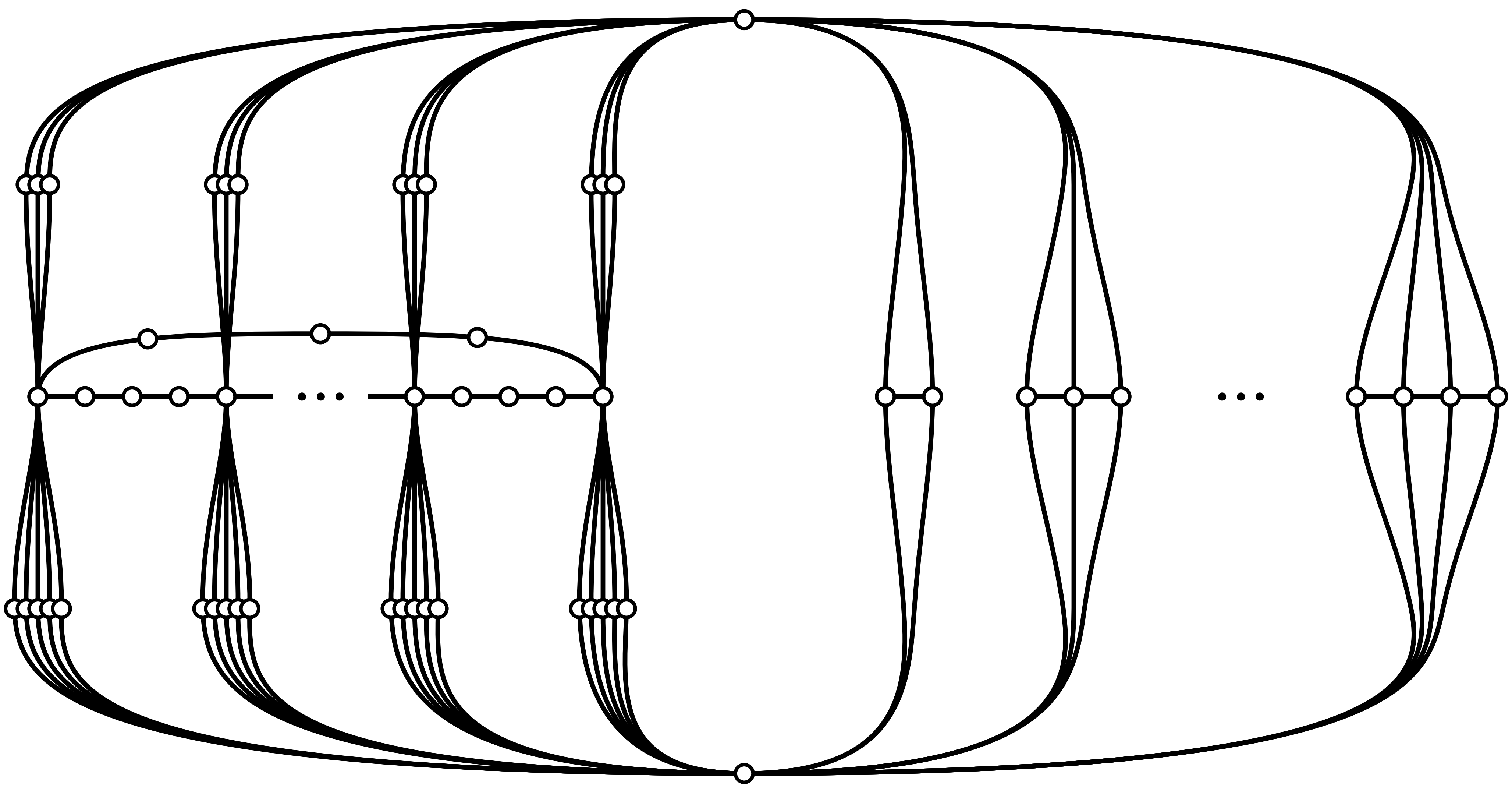


essentially, we have to solve UNARY BIN PACKING!

Type 1: Properties of Constructed Graphs

- near-planar graphs (:= planar + a single edge)
- fvs at most 3 (fvs := #vertices to delete to a forest)
- pathwidth at most 4
- distance to a path forest at most $O(b)$



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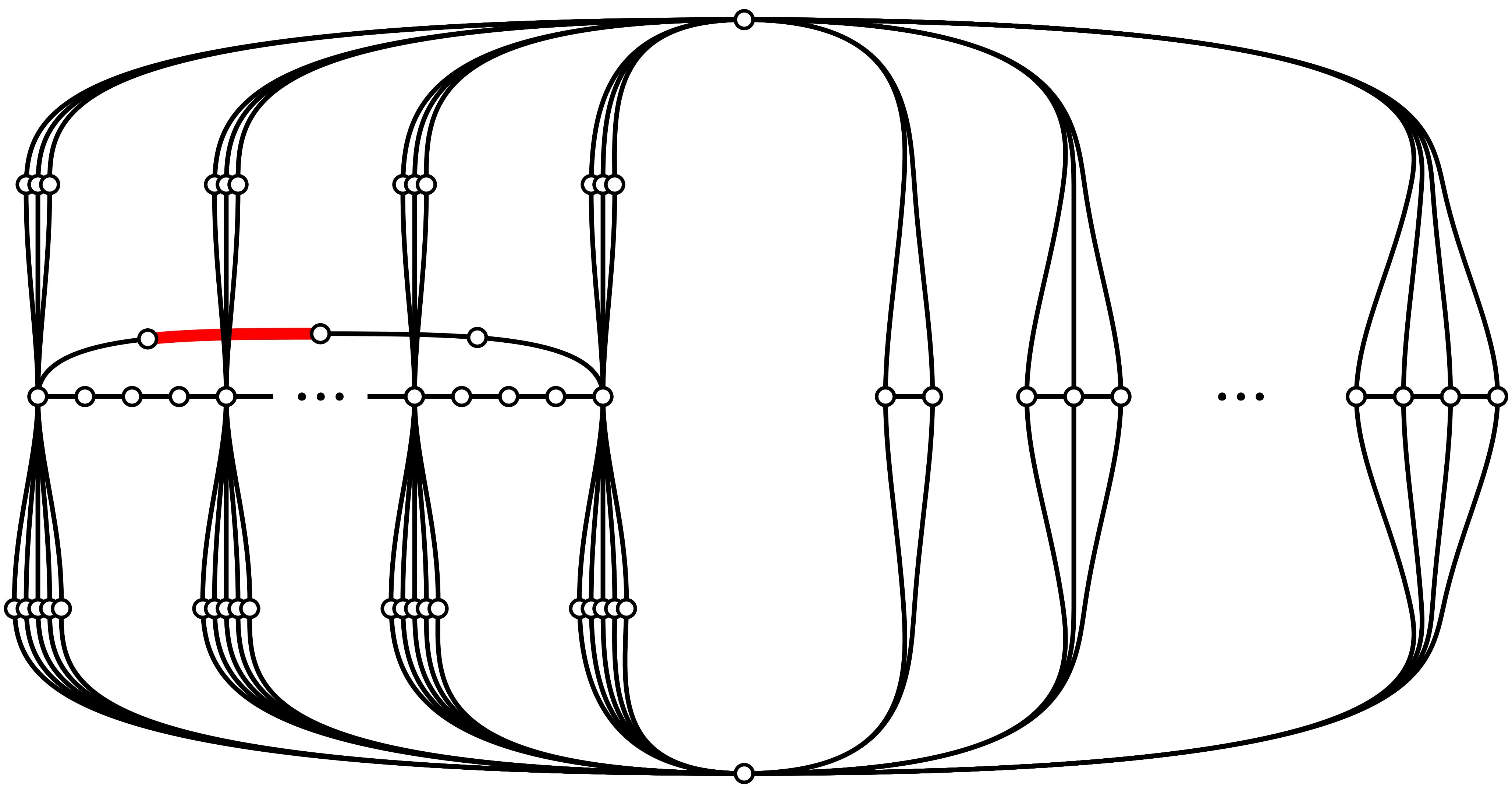

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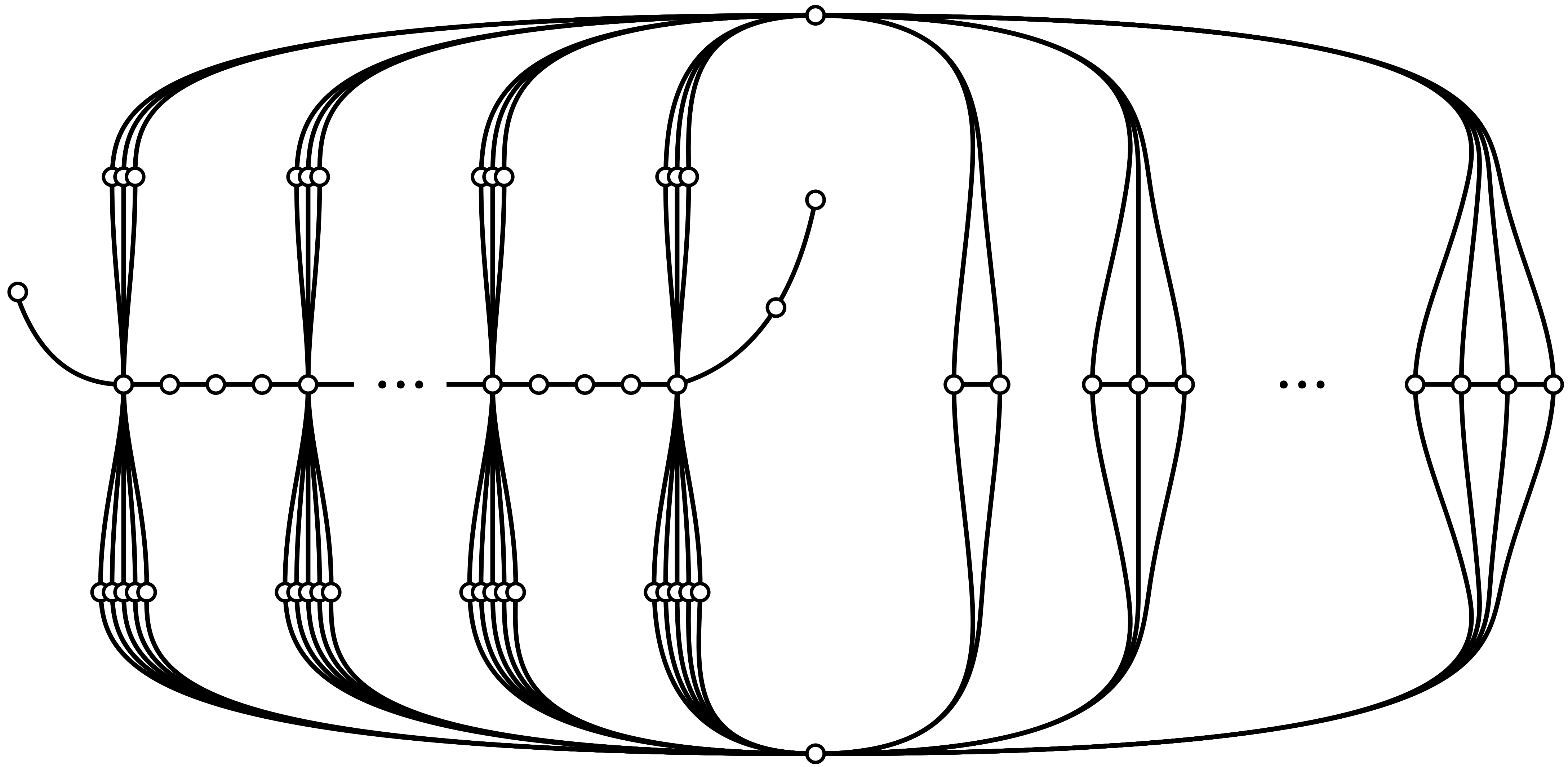

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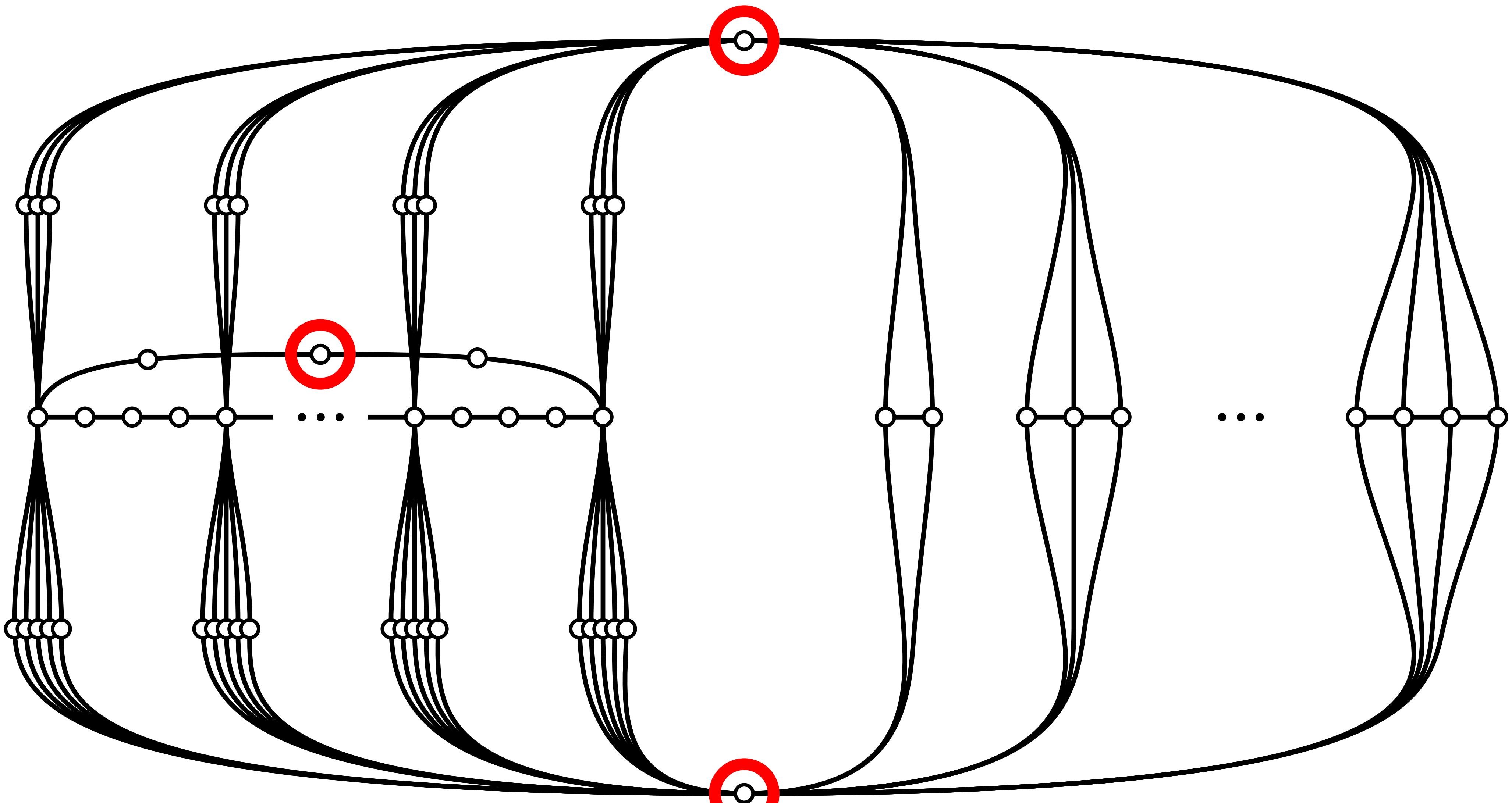

Type 1: Properties of Constructed Graphs

- near-planar graphs (:= planar + a single edge)
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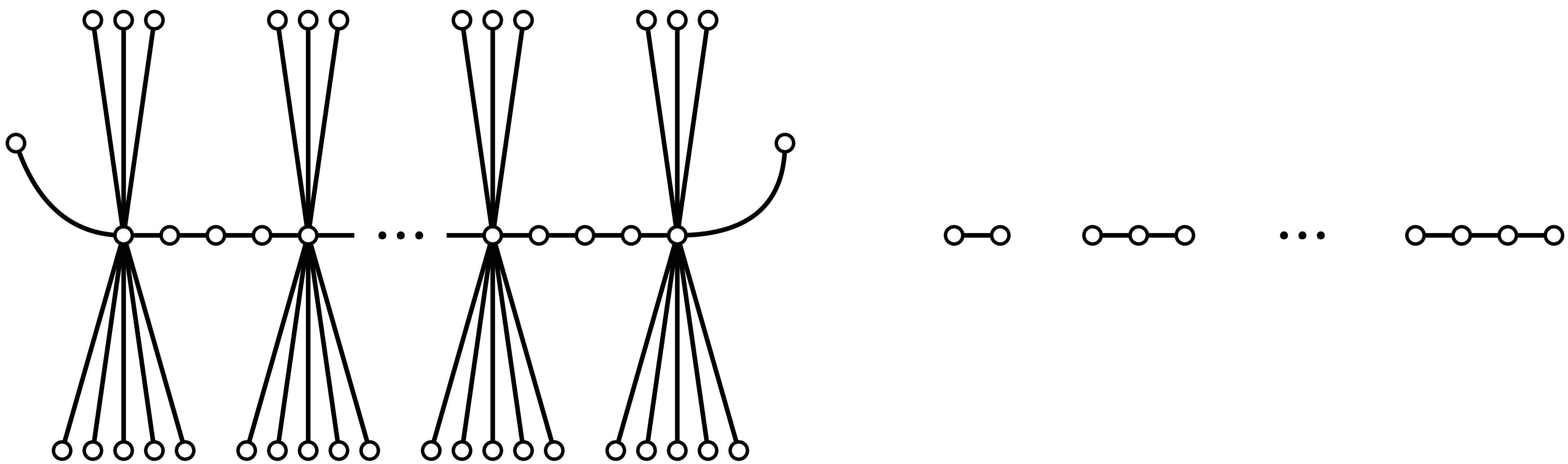
caterpillar



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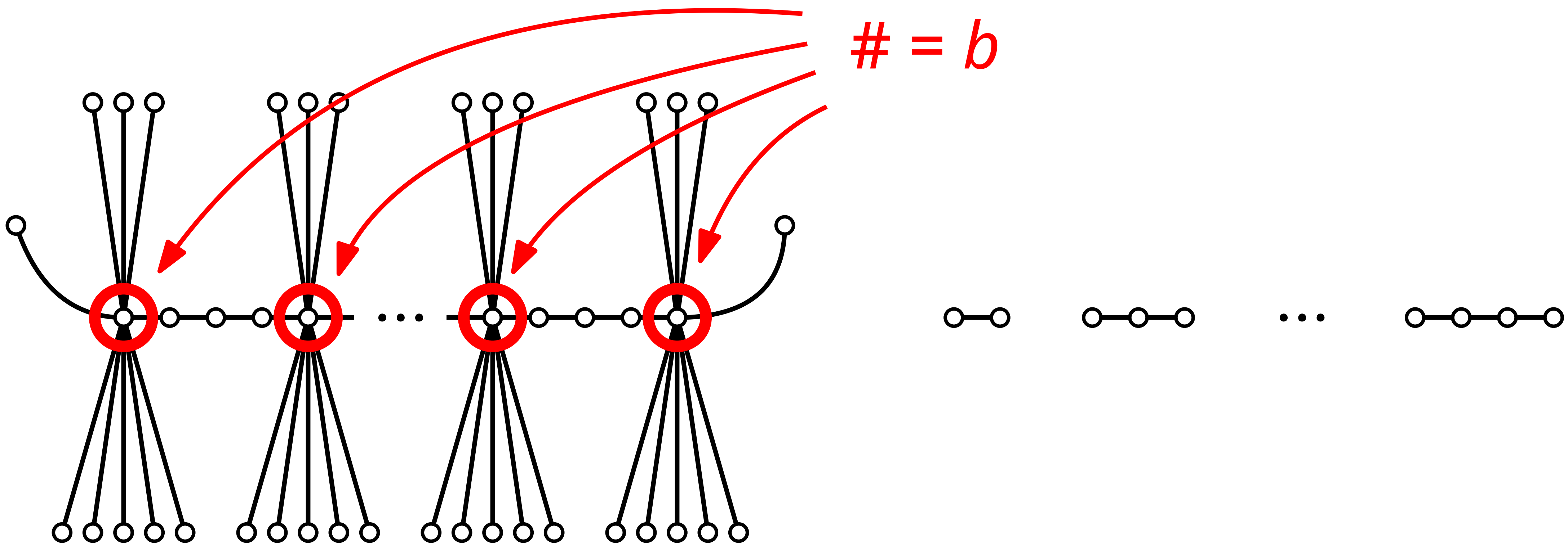

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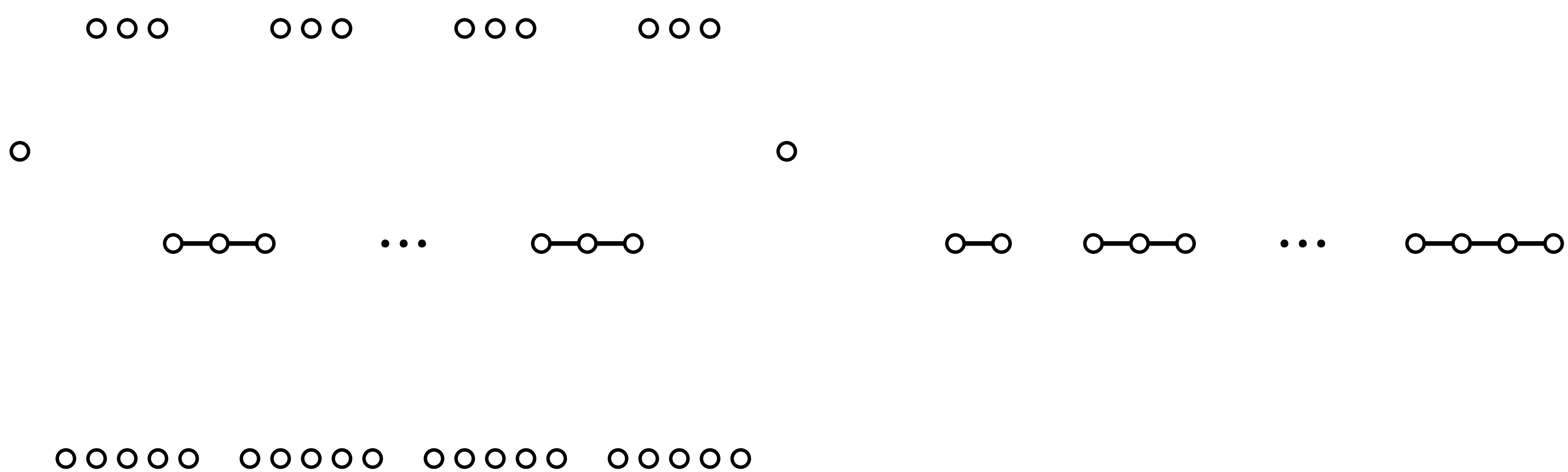


Type 1: Difference with [GB, 2007]

Type 1 is based on the first NP-hardness of 1-planarity.
[Grigoriev & Bodlaender, Algorithmica 2007]

- reduction from 3-PARTITION
  - → from UNARY BIN PACKING
    We can use W[1]-hard / b.
- unbounded fvs due to thick edges.

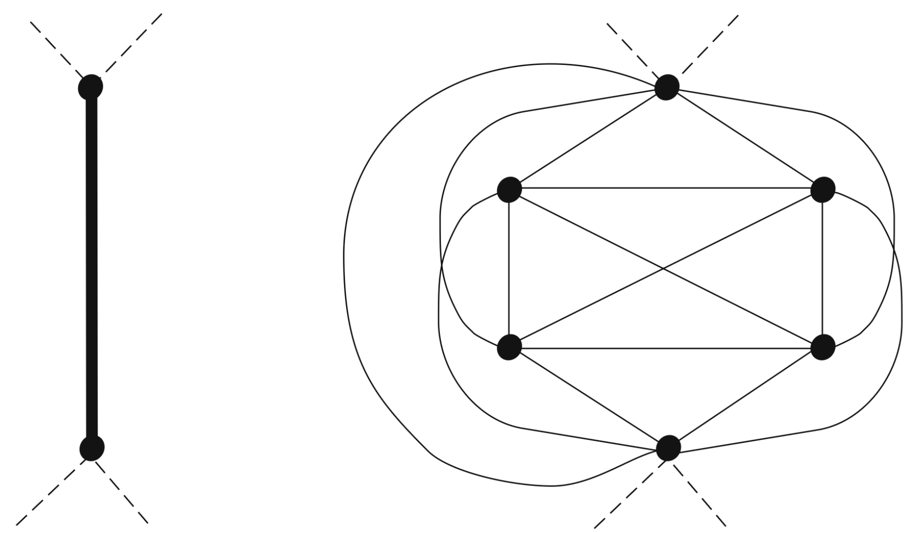
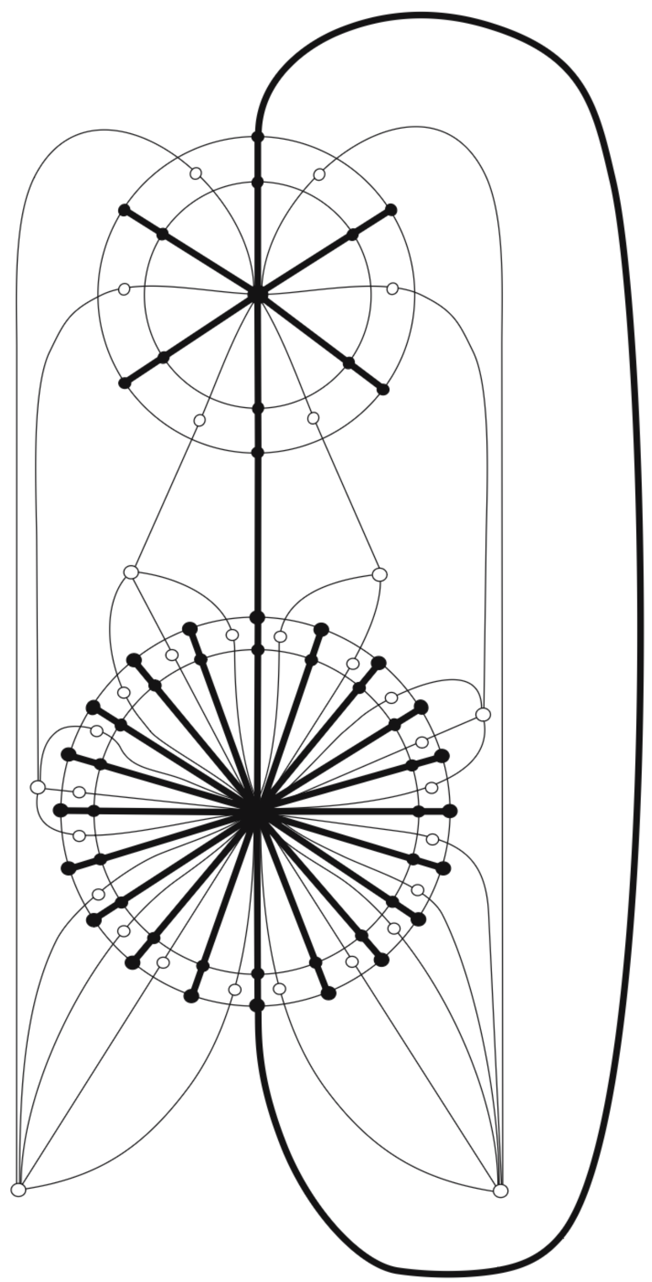


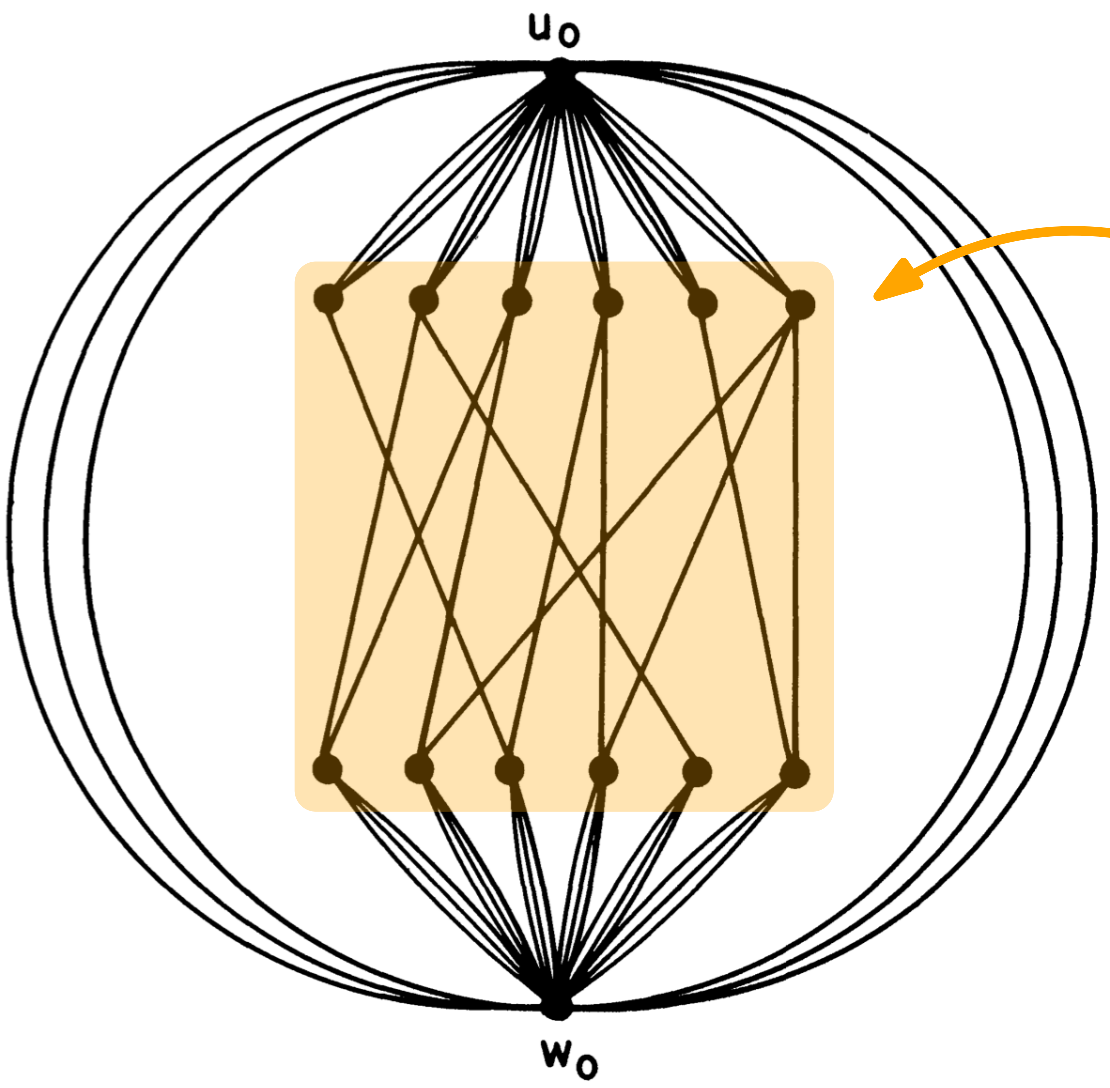
Fig. 1. Thick edge is a graph K_6



figures taken from the paper (https://doi.org/10.1007/11537311_33)

Type 2: Base Idea

For the other type of reduction,
we follow the idea of [Garey & Johnson, SIAM JADM 1983],
which showed the NP-hardness of CROSSING NUMBER.



an instance of
BIPARTITE CROSSING NUMBER

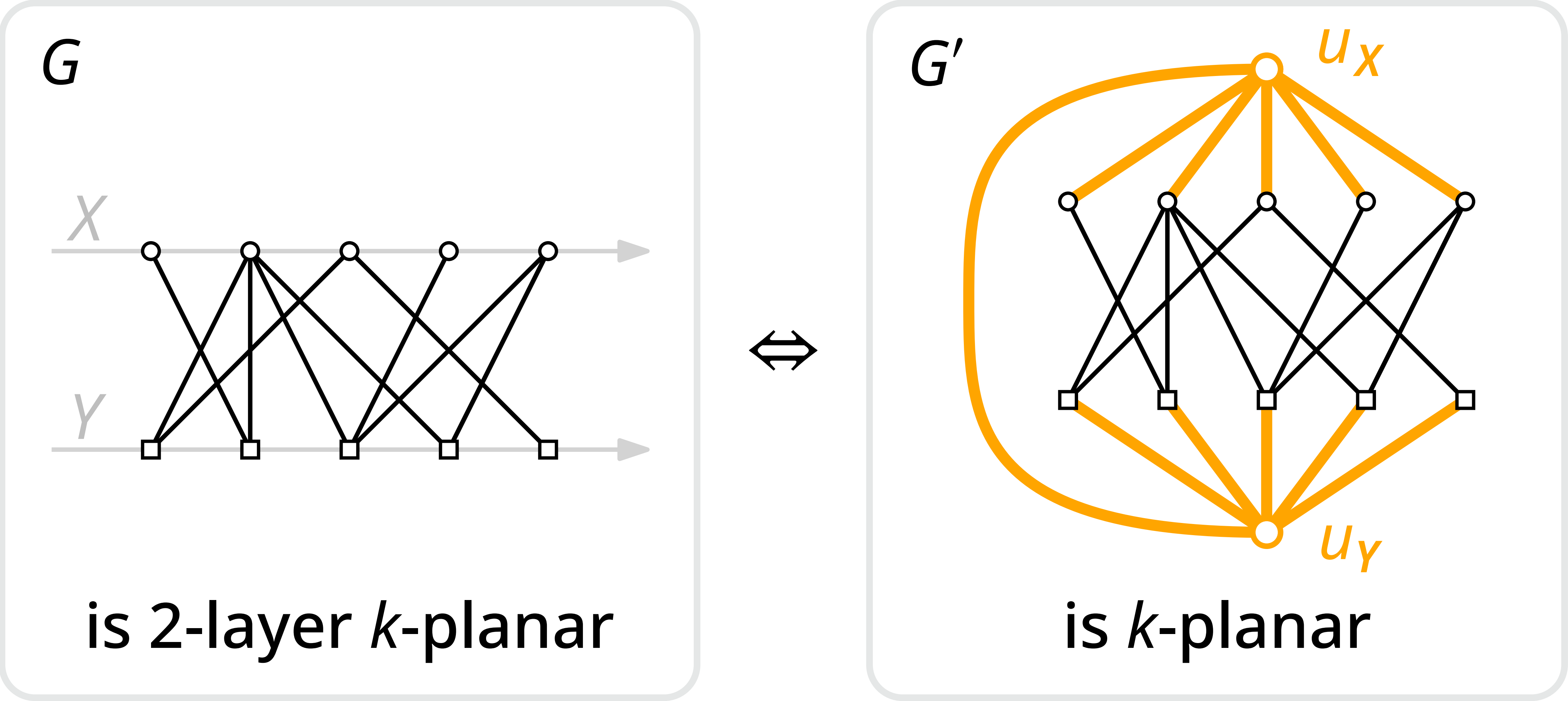
+

two additional vertices

figure taken from the paper (`https://doi.org/10.1137/0604033`)

Type 2: From 2-Layer k-Planarity

We do the same thing on k-Planarity!



Pipeline from Bandwidth

It propagates hardness of BANDWIDTH to k-PLANARITY.

BANDWIDTH

(fvs+=0) reduction

[K, O, Wolff, SoCG 2025]

2-LAYER k-PLANARITY

(fvs+=2) reduction

(in the previous page)

k-PLANARITY

on trees,

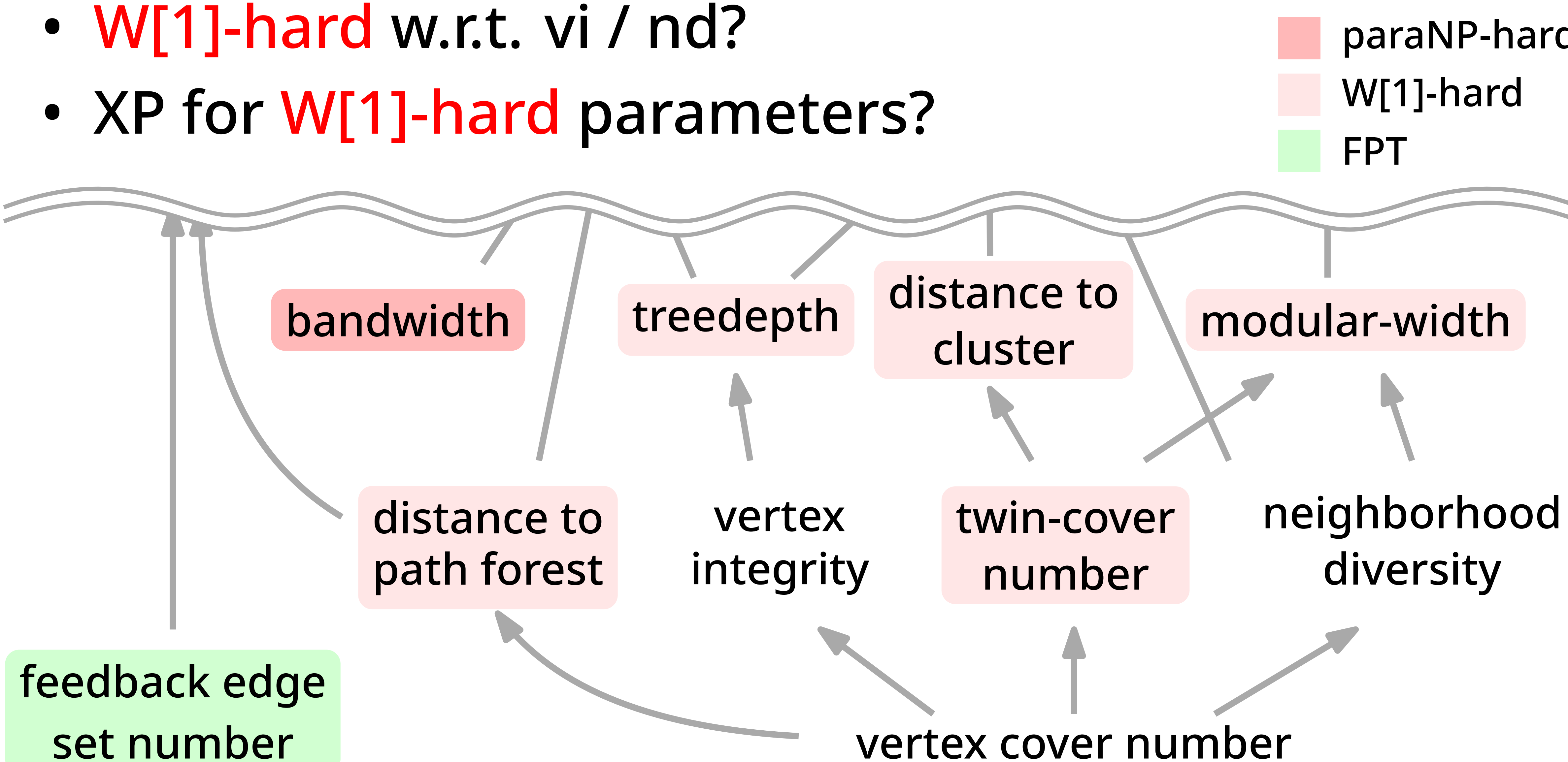
- no cosntant-factor approx.
- W[1]-hard w.r.t. treedepth

on **graphs with fvs=2**,

- no cosntant-factor approx.
- W[1]-hard w.r.t. treedepth

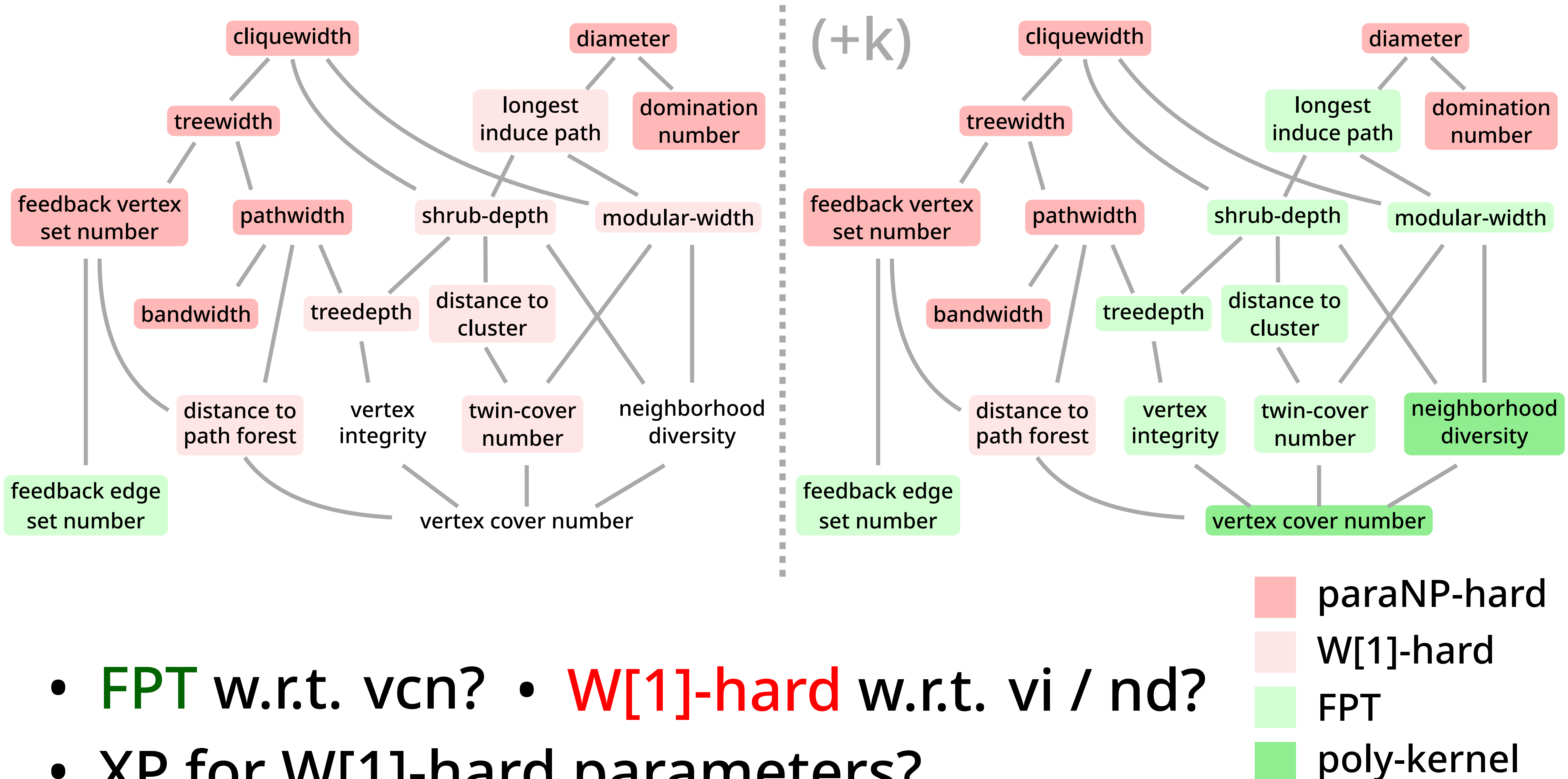
Open Problems

- FPT w.r.t. vertex cover number?
  - crossing minimization is FPT.
    [Hliněný & Sankaran, GD 2019]
- W[1]-hard w.r.t. vi / nd?
- XP for W[1]-hard parameters?



Summary

- 1-PLANARITY is NP-c / near-planar with pw$\leq$ 4, fvs $\leq$ 3.
- no constant-factor approximation for lcr(G).



- FPT w.r.t. vcn? • W[1]-hard w.r.t. vi / nd?
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